

Derivation of a Fundamental Diagram for Urban Traffic Flow

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Abstract. Despite the importance of urban traffic flows, there are only a few theoretical approaches to determine fundamental relationships between macroscopic traffic variables such as the traffic density, the utilization, the average velocity, and the travel time. In the past, empirical measurements have primarily been described by fit functions. Here, we derive expected fundamental relationships from a model of traffic flows at intersections, which suggest that the recently measured fundamental diagrams for urban flows can be systematically understood. In particular, this allows one to derive the average travel time and the average vehicle speed as a function of the utilization and/or the average number of delayed vehicles.

PACS. 89.40.Bb Land transportation – 47.10.ab Conservation laws and constitutive relations – 51.10.+y Kinetic and transport theory of gases

1 Introduction

The study of urban traffic flows has a long history (see Ref. [1] for an overview). For more than a decade now, physicists have contributed various interesting models, ranging from cellular automata [2–4] to fluid-dynamic approaches [5,6]. Complementary, one should mention, for example, Refs. [7,8] as representatives of publications by traffic engineers, and also continuous microscopic flow models used in commercial software tools such as VISSIM.

One research area in traffic physics is the transition from free to congested traffic in urban road networks [9, 10], which started off with the paper by Biham, Middleton and Levine [11]. Interestingly enough, the spreading of congestion seems to share some features with cascading failures [12].

In the following, we will focus on the study of fundamental relationships between flow, utilization, and density on the one hand and the average velocity or travel time on the other hand. Such relationships were already studied in the 60ies [13–23], but this work primarily took a phenomenological approach. Moreover, the recorded data in the regime of congested road networks scattered enormously. Thus, it was hard to fit a curve. Consequently, many different relationships were proposed. The probably most wide-spread formula is the one published by the Bureau of Public Roads [16]. Accordingly, the travel time \mathcal{T}_i of an urban road section i with capacity C_i would follow the capacity constraint function

$$\mathcal{T}_i = \mathcal{T}_i^0 \left[1 + \alpha_i \left(\frac{A_i}{C_i} \right)^{\beta_i} \right]. \quad (1)$$

(see Fig. 5). Here, A_i is the arrival flow in that road section and \mathcal{T}_i^0 the travel time for light traffic conditions,

while $\alpha_i \approx 0.5$ and $\beta_i \approx 4$ are fit parameters. It is obvious that this formula does not diverge when the capacity C_i is reached, as one may expect. This is just one of the many theoretical inconsistencies of the proposed phenomenological formulas. It is no wonder that the subject of fundamental diagrams for urban traffic has recently been taken up again [24–31].

Note that even Robert Herman and nobel prize laureate Ilya Prigogine addressed traffic flow in cities [32,33]. They tried to derive fundamental relationships via a statistical physics approach, which however still contained phenomenological elements. Their “two-fluid approach” considered moving and standing traffic and the mutual interdependencies between them.

Recently, the issue of urban gridlock was reanimated by Carlos Daganzo [28]. Together with Geroliminis, he presented convincing evidence for a fundamental diagram of urban traffic flow [29,30]. Before, fundamental diagrams were mainly used for the study of freeway systems [34]. There, they were very useful to understand capacity effects [1], different traffic states [35], and the traffic dynamics, in particular the propagation of shock fronts [36]. It is, therefore, not surprising that people are eager to find fundamental relationships for urban traffic as well.

After discussing elementary relationships for cyclically signalized intersections of urban road networks in Sec. 2, we will start in Sec. 3 with the discussion of *undersaturated* traffic conditions and derive a relationship for the average travel time as a function of the utilization or the number of delayed vehicles. Furthermore, we will determine a formula for the average speed. In Sec. 4, we will extend the analysis to *congested* road conditions, where the intersection capacity is exceeded. Afterwards, in Sec. 5, we will indicate how to deal with *oversaturated* networks, where

the link capacity is insufficient to take up all vehicle that would like to enter a road section. Finally, Sec. 6 provides a summary and discussion.

2 Elementary Relationships for Cyclically Operated Intersections

Let us study a single intersection with a periodically operated traffic light. We shall have green phases j of duration ΔT_j , during which one or several of the traffic streams i are served. β_{ij} shall be 1, if traffic stream i is served by green phase j , otherwise $\beta_{ij} = 0$. The setup time after phase j shall require a time period τ_j . It may be imagined to correspond to the time period of the amber light (although in practice, this has to be corrected for reaction times and intersection clearing times). The sum of setup times will be called the “lost service time”

$$T_{\text{los}} = \sum_j \tau_j, \quad (2)$$

while the sum of all green time periods and setup times will be called the “cycle time”

$$T_{\text{cyc}} = \sum_j (\Delta T_j + \tau_j) = T_{\text{los}} + \sum_j \Delta T_j. \quad (3)$$

The green times are also sometimes expressed as fractions $f_j \geq 0$ of the cycle time, i.e.

$$\Delta T_j = f_j T_{\text{cyc}} \quad (4)$$

with $\sum_j f_j < 1$. After inserting this into Eq. (3) and rearranging terms, we get

$$T_{\text{cyc}}(\{f_j\}) = \frac{T_{\text{los}}}{1 - \sum_j f_j}, \quad (5)$$

i.e. the cycle time is proportional to the lost service time T_{los} .

Assuming average inflows A_i per lane, the number of vehicles belonging to traffic stream i , which must be served within one cycle time T_{cyc} , amounts to $A_i T_{\text{cyc}}$. In order to avoid the formation of growing queues (i.e. the onset of congestion), the number of served vehicles per cycle time and lane must reach this value. The number of vehicles of traffic stream i potentially served during the green phases ΔT_j is given by $\sum_j Q_{ij} \beta_{ij} \Delta T_j$ per lane, where Q_{ij} denotes the outflow capacity (discharge flow) per lane, when stream i is served by phase j . If we want to have a certain amount of excess capacity to cope with a variability of the inflows, we may demand

$$(1 + \delta_i) A_i T_{\text{cyc}} = \sum_j Q_{ij} \beta_{ij} \Delta T_j = \sum_j Q_{ij} \beta_{ij} f_j T_{\text{cyc}} \quad (6)$$

with $\delta_i > 0$. This linear set of equations may be solved for the green time fractions f_j . In the following, we will assume the case, where not *several* traffic streams i are

served in parallel by one and the same green phase, but where each phase j serves a single traffic stream i . Then, we may assume $\beta_{ij} = 1$, if $j = i$, and $\beta_{ij} = 0$ otherwise. Furthermore, $Q_{ij} = Q_{ii} = \widehat{Q}_i$. With this, Eq. (6) implies

$$f_i(u_i, \delta_i) = (1 + \delta_i) \frac{A_i}{\widehat{Q}_i} = (1 + \delta_i) u_i, \quad (7)$$

where

$$u_i = \frac{A_i}{\widehat{Q}_i} \quad (8)$$

is called the *utilization* of the service or outflow capacity \widehat{Q}_i (see Fig. 1). $\delta_i \geq 0$ is a safety factor to cope with variations in the arrival flow (inflow) A_i . According to Eq. (5) we must have

$$0 < 1 - \sum_i f_i = 1 - \sum_i (1 + \delta_i) \frac{A_i}{\widehat{Q}_i}. \quad (9)$$

Otherwise, we will have growing vehicle queues from one signal cycle to the next, corresponding to the congested traffic regime.

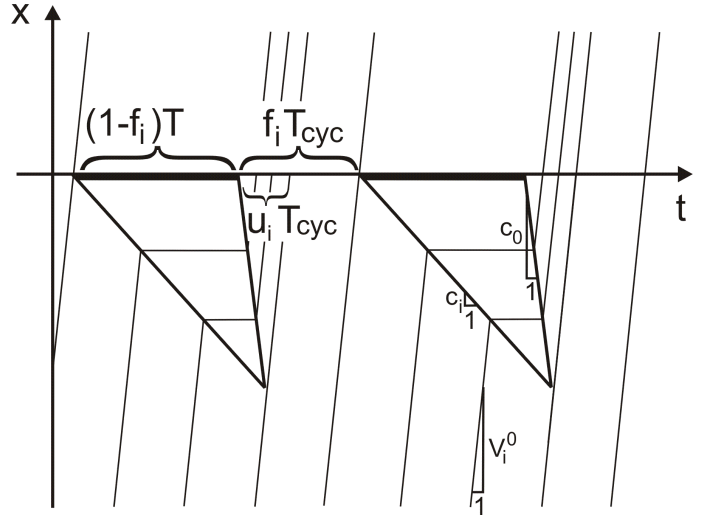


Fig. 1. Illustration of vehicle trajectories for a traffic light, which has an amber and red time period of altogether $(1 - f_i)T_{\text{cyc}}$ and a green time period of $f_i T_{\text{cyc}}$. Vehicles move forward at the free speed V_i^0 or are stopped in a vehicle queue (horizontal lines), which forms during the amber and red time period behind the traffic light (located at the t -axis). The speed of the upstream moving congestion front is given by the arrival flow [36] and denoted by $c_i \leq 0$. The dissolution speed $c_0 < 0$ of congested traffic is a characteristic constant with $|c_0| \geq |c_i|$ [6, 37]. The average delay time can be determined by averaging over the waiting times in the triangular areas. Note that for the case of an excess greentime ($f_i > u_i$), vehicles may pass the traffic light without any delay.

Note that, at time t , the number $N_i(t)$ of vehicles per lane in the road section reserved for traffic stream i is

given by the time integral of the arrival flow $A_i(t)$ minus the departure flow $\gamma_i(t)O_i(t)$:

$$N_i(t) = \int_{t_0}^t dt' [A_i(t') - \gamma_i(t')O_i(t')]. \quad (10)$$

Here, the starting time t_0 must be properly chosen to give the correct initial number of vehicles on road section i . During the amber and red time periods, we set the permeability $\gamma_i(t) = 0$, as there is no outflow, while $\gamma_i(t) = 1$ during green phases. The departure flow O_i per lane is given by the service capacity \widehat{Q}_i per lane, as long as there is a positive number $\Delta N_i(t) > 0$ of delayed vehicles. If Q_{out} represents the characteristic outflow from congested traffic per lane into an area of free flow, the *overall* service capacity by *all* service lanes is given by the minimum of the number of lanes I_i used by vehicle stream i upstream of the intersection, and the number I'_i of lanes downstream of it:

$$I_i \widehat{Q}_i = \min(I_i, I'_i) Q_{\text{out}}, \quad \text{i.e.} \quad \widehat{Q}_i = \min\left(1, \frac{I'_i}{I_i}\right) Q_{\text{out}}. \quad (11)$$

When the vehicle queue forming behind a traffic light has completely resolved, the outflow from the road section used by vehicle stream i drops from \widehat{Q}_i to a lower value. Then, if the greentime period continues, the outflow $O_i(t)$ per lane corresponds to the arrival flow $A_i(t - T_i^0)$ per lane expected at the end of road section i under free flow conditions [37]. Here, $T_i^0 = L_i/V_i^0$ represents the travel time under free flow conditions, which is obtained by division of the length L_i of the road section reserved for stream i by the free speed V_i^0 . Considering also the above definition of the permeabilities $\gamma_i(t)$ reflecting the time-dependent states of the traffic signal, we have

$$\gamma_i(t)O_i(t) = \gamma_i(t) \begin{cases} \widehat{Q}_i & \text{if } \Delta N_i(t) > 0, \\ A_i(t - T_i^0) & \text{otherwise.} \end{cases} \quad (12)$$

The number $\Delta N_i(t)$ of *delayed* vehicles per lane at time t on the road section reserved for vehicle stream i can be easily determined as well. In contrast to Eq. (10), we have to subtract the integral of the departure flow from the integral of the arrival flow $A_i(t - T_i^0)$ expected at the *end* of road section i under free flow conditions. Altogether, the number of delayed vehicles can be calculated as

$$\Delta N_i(t) = \int_{t_0}^t dt' [A_i(t' - T_i^0) - \gamma_i(t')O_i(t')]. \quad (13)$$

If the traffic flow is organized as a vehicle platoon and the green phase is synchronized with its arrival at the traffic light, the number of delayed vehicles is zero. However, if the traffic flow A_i is uniform, we find

$$\Delta N_i(t) = A_i \cdot (t - t_0) - \int_{t_0}^t dt' \gamma_i(t')O_i(t'). \quad (14)$$

Let us assume that t_0 denotes the time when the green phase for traffic stream i ended. Then, the next green phase for this traffic stream starts at time $t'_0 = t_0 + (1 - f_i)T_{\text{cyc}}$, as $f_i T_{\text{cyc}}$ is the green time period and $(1 - f_i)T_{\text{cyc}}$ amounts to the sum of the amber and red time periods. Due to $O_i(t) \geq A_i(t)$ and $O_i(t'_0) > A_i(t'_0)$, $t'_0 - t_0 = (1 - f_i)T_{\text{cyc}}$ is also the time period after which the maximum number ΔN_i^{max} of delayed vehicles is reached.

In case of a uniform arrival of vehicles at the rate $A_i = u_i \widehat{Q}_i$ per lane we have

$$\begin{aligned} \Delta N_i^{\text{max}}(u_i, \{f_j\}) &= A_i(1 - f_i)T_{\text{cyc}}(f_i) \\ &= u_i \widehat{Q}_i(1 - f_i)T_{\text{cyc}}(\{f_j\}). \end{aligned} \quad (15)$$

Since $\widehat{Q}_i - A_i$ is the rate at which this vehicle queue can be reduced (considering the further uniform arrival of vehicles at rate A_i), it takes a green time period of

$$T_i(u_i, \{f_j\}) = \frac{A_i(1 - f_i)T_{\text{cyc}}}{\widehat{Q}_i - A_i} = \frac{u_i(1 - f_i)}{1 - u_i} T_{\text{cyc}}(\{f_j\}), \quad (16)$$

until this vehicle queue is again fully resolved, and newly arriving vehicles can pass the traffic light without any delay.

Finally, let us determine the average delay time of vehicles. If we have a platoon of vehicles which is served by a properly synchronized traffic light, the average delay is $T_i^{\text{av}} = T_i^{\text{min}} = 0$. However, if we have a constant arrival flow A_i , the average delay T_i^{av} of queued vehicles is given by the arithmetic mean $(T_i^{\text{max}} + T_i^{\text{min}})/2$ of the maximum delay for the first vehicle in the queue behind the traffic light, which corresponds to the amber plus red time period

$$T_i^{\text{max}}(\{f_j\}) = (1 - f_i)T_{\text{cyc}}(\{f_j\}), \quad (17)$$

and the minimum delay $T_i^{\text{min}} = 0$ of a vehicle arriving just at the time when the queue is fully dissolved. To get the average delay, we have to weight this by the percentage of delayed vehicles. While the number of vehicles arriving during the cycle time T_{cyc} is $A_i T_{\text{cyc}}$, the number of undelayed vehicles is given by $A_i(\Delta T_i - T_i)$. Considering formulas (4) and (16), the excess green time is

$$\Delta T_i - T_i = f_i T_{\text{cyc}} - \frac{u_i(1 - f_i)T_{\text{cyc}}}{1 - u_i} = \frac{f_i - u_i}{1 - u_i} T_{\text{cyc}}. \quad (18)$$

Hence, the percentage of delayed vehicles is

$$\frac{A_i[T_{\text{cyc}} - (\Delta T_i - T_i)]}{A_i T_{\text{cyc}}} = 1 - \frac{f_i - u_i}{1 - u_i} = \frac{1 - f_i}{1 - u_i} \leq 1. \quad (19)$$

Altogether, the average delay of *all* vehicles is expected to be

$$T_i^{\text{av}}(u_i, \{f_j\}) = \frac{(1 - f_i)}{(1 - u_i)} \frac{T_i^{\text{max}}(\{f_j\})}{2} = \frac{(1 - f_i)^2}{1 - u_i} \frac{T_{\text{cyc}}(\{f_j\})}{2} \quad (20)$$

Inserting Eq. (5) gives

$$T_i^{\text{av}}(u_i, \{f_j\}) = \frac{(1 - f_i)^2}{(1 - u_i)} \frac{T_{\text{los}}}{2(1 - \sum_j f_j)}, \quad (21)$$

while with Eq. (15) we obtain

$$\mathcal{T}_i^{\text{av}}(u_i, \{f_j\}) = \frac{1 - f_i}{1 - u_i} \frac{\Delta N_i^{\text{max}}(u_i, \{f_j\})}{2u_i \widehat{Q}_i}. \quad (22)$$

Therefore, the average delay time is proportional to the maximum queue length ΔN_i^{max} , but the prefactor depends on $f_i = (1 + \delta_i)u_i$ (or the safety factor δ_i , respectively). In case of no excess green time ($\delta_i = 0$ and $f_i = u_i$), we just have

$$\begin{aligned} \mathcal{T}_i^{\text{av}}(\{u_j\}) &= (1 - u_i) \frac{T_{\text{cyc}}(\{u_j\})}{2} = \frac{\Delta N_i^{\text{max}}(\{u_j\})}{2A_i} \\ &= \frac{\Delta N_i^{\text{max}}(\{u_j\})}{2u_i \widehat{Q}_i} \end{aligned} \quad (23)$$

(see Fig. 2).

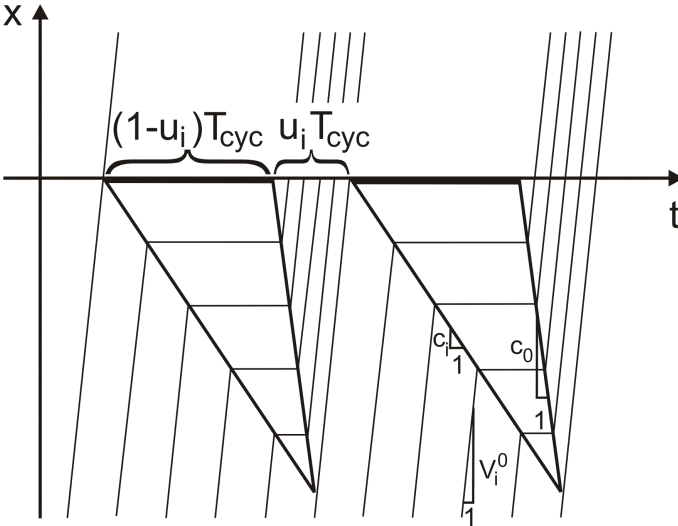


Fig. 2. Illustration of vehicle trajectories and signal operation in the case $f_i = u_i$, where there are no excess green times so that the traffic light is turned red as soon as the vehicle queue has been fully resolved.

Similarly to the average travel time, we may determine the average queue length. As the average number of delayed vehicles is $(\Delta N_i^{\text{max}} + 0)/2$ and a fraction $(1 - f_i)/(1 - u_i) \leq 1$ of vehicles is delayed, together with Eqs. (15) and (20) we find

$$\begin{aligned} \Delta N_i^{\text{av}}(u_i, \{f_j\}) &= \frac{(1 - f_i)}{(1 - u_i)} \frac{\Delta N_i^{\text{max}}(u_i, \{f_j\})}{2} \\ &= u_i \widehat{Q}_i \frac{(1 - f_i)^2 T_{\text{cyc}}(\{f_j\})}{(1 - u_i)^2} \\ &= u_i \widehat{Q}_i \mathcal{T}_i^{\text{av}}(u_i, \{f_j\}). \end{aligned} \quad (24)$$

This remarkably simple relationship is known in queuing theory as Little's Law [38], which holds for time-averaged variables even in the case of non-uniform arrivals, if the

system behaves stable (i.e. the queue length is not *systematically* growing or shrinking). It also allows one to establish a direct relationship between the “average vehicle density”

$$\rho_i^{\text{av}} = \frac{\Delta N_i^{\text{av}}}{L_i} \quad (25)$$

on the road section used by stream i and the utilization $u_i = A_i/\widehat{Q}_i$, namely

$$\rho_i^{\text{av}} = \frac{u_i \widehat{Q}_i \mathcal{T}_i^{\text{av}}}{L_i}. \quad (26)$$

Note that the average density ρ_i^{av} has the same dependencies on other variables as ΔN_i^{av} or $\mathcal{T}_i^{\text{av}}$, and an additional dependency on L_i , i.e. the more natural quantity to use is the queue length ΔN_i^{av} .

2.1 Efficiency of Traffic Operation

In reality, the average delay time will depend on the time-dependence of the inflow $A_i(t)$, and on how well the traffic light is coordinated with the arrival of vehicle platoons. In the best case, the average delay is zero, but in the worst case, it may also be larger than

$$\frac{1 - u_i}{2} T_{\text{cyc}}(\{u_j\}) = \frac{(1 - u_i) T_{\text{los}}}{2(1 - \sum_j u_j)}, \quad (27)$$

see Eq. (20) with $f_i = u_i$. We may, therefore, introduce an efficiency coefficient ϵ_i by the definition

$$\mathcal{T}_i^{\text{av}}(\{u_j\}, \epsilon_i) = (1 - \epsilon_i) \frac{(1 - u_i) T_{\text{los}}}{2(1 - \sum_j u_j)} \quad (28)$$

or, considering Eq. (24), equivalently by

$$\Delta N_i^{\text{av}}(\{u_j\}, \epsilon_i) = (1 - \epsilon_i) u_i \widehat{Q}_i \frac{(1 - u_i) T_{\text{los}}}{2(1 - \sum_j u_j)}. \quad (29)$$

For $\epsilon_i = 1$, the traffic light is perfectly synchronized with platoons in traffic stream i , i.e. vehicles are served without any delay, while for $\epsilon_i = 0$, the delay corresponds to uniform arrivals of vehicles, when no excess green time is given. If the traffic light is not well synchronized with the arrival of vehicle platoons, we may even have $\epsilon_i < 0$. It also makes sense to define an efficiencies not only for the traffic phases, but also for the operation of a traffic light (i.e. the full cycle). This can be done by averaging over all efficiencies ϵ_i (and potentially weighting them by the number $u_i \widehat{Q}_i T_{\text{cyc}}$ of vehicles arriving in one cycle). Therefore, it makes sense to define the intersection efficiency as

$$\epsilon = \frac{\sum_i \epsilon_i u_i \widehat{Q}_i}{\sum_i u_i \widehat{Q}_i}. \quad (30)$$

Note that, particularly in cases of pulsed rather than uniform arrivals of vehicles, the efficiencies ϵ_i depend on the cycle time T_{cyc} and, therefore, also on the utilizations u_i .

Increasing the efficiency ϵ_i for one traffic stream i will often (but not generally) reduce the efficiency ϵ_j of another traffic stream j , which poses a great challenge to traffic optimization.

In case of excess green times ($\delta_i > 0$) and uniform vehicle arrivals, one may derive the related efficiency from Eqs. (21) and (28). We obtain

$$1 - \epsilon_i = \frac{(1 - f_i)^2}{(1 - u_i)^2} \frac{(1 - \sum_j u_j)}{(1 - \sum_j f_j)}, \quad (31)$$

where $f_i(u_i, \delta_i) = (1 + \delta_i)u_i$ according to Eq. (7). The efficiency ϵ_i is usually smaller than in the case, where the traffic light is turned red as soon as a vehicle queue has been dissolved (for exceptions see Ref. [39]). Then, $\epsilon_i < 0$ for $\delta_i > 0$. Formula (31) also allows one to treat the case where the green time fractions f_i and the cycle time T_{cyc} are not adapted to the respective traffic situation, but where a fixed cycle time $T_{\text{cyc}} = T_{\text{cyc}}^0$ and fixed green time fractions f_i^0 are implemented. This corresponds to constant green times

$$f_i^0 T_{\text{cyc}}^0(\{f_j^0\}) = \frac{f_i^0 T_{\text{los}}}{1 - \sum_j f_j^0}. \quad (32)$$

In case of uniform vehicle arrivals, we just have to insert the corresponding value $f_i = f_i^0$ into Eq. (31) to obtain ϵ_i . In the case of non-uniform arrivals, ϵ_i can be understood as fit parameter of our model, which allows us to adjust our formulas to empirical data and to quantify the efficiency of traffic light operation. In this way, we can also absorb effects of stochastic vehicle arrivals into the efficiency coefficients ϵ_i , which simplifies our treatment a lot.

3 Fundamental Relationships for Undersaturated Traffic

The travel time is generally given by the sum of the free travel time $T_i^0 = L_i/V_i^0$ and the average delay time T_i^{av} , where L_i denotes the length of the road section used by vehicle stream i and V_i^0 the free speed (or speed limit). With Eq. (24), we get

$$T_i(\{u_j\}, \epsilon_i) = T_i^0 + T_i^{\text{av}}(\{u_j\}, \epsilon_i) = \frac{L_i}{V_i^0} + \frac{\Delta N_i^{\text{av}}(\{u_j\}, \epsilon_i)}{u_i \widehat{Q}_i} \quad (33)$$

Inserting Eq. (29), we can express the travel time solely in terms of the utilization u_i , and we have

$$T_i(\{u_j\}, \epsilon_i) = \frac{L_i}{V_i^0} + (1 - \epsilon_i) \frac{(1 - u_i) T_{\text{los}}}{2(1 - \sum_j u_j)}. \quad (34)$$

The formula (34) constitutes a fundamental relationship between the average travel time T_i^{av} on the capacity utilization u_i under the assumptions made (mainly cyclical operation with certain efficiencies ϵ_i). Of course, one still needs to specify the factor $(1 - \epsilon_i)$. In case of constant

arrival rates A_i , this factor is given by Eq. (31), which finally results in

$$T_i(u_i, \{f_j\}) = \frac{L_i}{V_i^0} + \frac{(1 - f_i)^2}{(1 - u_i)} \frac{T_{\text{los}}}{2(1 - \sum_j f_j)}. \quad (35)$$

After insertion of Eq. (7), we get

$$T_i(\{u_j\}, \{\delta_j\}) = \frac{L_i}{V_i^0} + \frac{[1 - (1 + \delta_i)u_i]^2 T_{\text{los}}}{(1 - u_i) 2[1 - \sum_j (1 + \delta_j)u_j]}. \quad (36)$$

Sometimes, it is desirable to express the fundamental relationships in terms of the density rather than the utility. Inserting Eq. (34) into $T_i^{\text{av}}(u_i, \epsilon_i) = T_i(u_i, \epsilon_i) - L_i/V_i^0$, and this into Eq. (26), we obtain the equation

$$\rho_i^{\text{av}}(\{u_j\}, \epsilon_i, L_i) = \frac{u_i \widehat{Q}_i}{L_i} (1 - \epsilon_i) \frac{(1 - u_i) T_{\text{los}}}{2(1 - \sum_j u_j)}, \quad (37)$$

which can be numerically inverted to give the utilization u_i as a function of the scaled densities $\rho_j^{\text{av}} L_j / (1 - \epsilon_j)$. The calculations are simpler in case of a fixed cycle time T_{cyc}^0 and an uniform arrival of vehicles. By inserting Eq. (20) into (26) [and with $f_i = f_i^0$, $T_{\text{cyc}} = T_{\text{cyc}}^0$, see Eq. (32)], we get

$$\begin{aligned} L_i \rho_i^{\text{av}}(u_i, \{f_j^0\}) &= \Delta N_i^{\text{av}}(u_i, \{f_j^0\}) \\ &= u_i \widehat{Q}_i \frac{(1 - f_i^0)^2 T_{\text{cyc}}^0(\{f_j^0\})}{2(1 - u_i)}, \end{aligned} \quad (38)$$

which finally yields

$$u_i(\rho_i^{\text{av}} L_i, \{f_j^0\}) = \left(1 + (1 - f_i^0)^2 \frac{\widehat{Q}_i T_{\text{cyc}}^0(\{f_j^0\})}{2 \rho_i^{\text{av}} L_i} \right)^{-1}. \quad (39)$$

This can be inserted into Eq. (26) to give

$$\begin{aligned} T_i^{\text{av}}(\rho_i^{\text{av}} L_i, \{f_j^0\}) &= \frac{\rho_i^{\text{av}} L_i}{u_i(\rho_i^{\text{av}} L_i, \{f_j^0\}) \widehat{Q}_i} \\ &= \frac{\rho_i^{\text{av}} L_i}{\widehat{Q}_i} + (1 - f_i^0)^2 \frac{T_{\text{cyc}}^0(\{f_j^0\})}{2}. \end{aligned} \quad (40)$$

3.1 Transition to Congested Traffic

The utilizations u_i increase proportionally to the arrival flows A_i , i.e. they go up during the rush hour. Eventually,

$$\sum_j f_j = \sum_j (1 + \delta_j) u_j \rightarrow 1, \quad (41)$$

which means that the intersection capacity is reached. Sooner or later, there will be no excess capacities anymore, which implies $\delta_i \rightarrow 0$ and $f_i \rightarrow u_i$. In this case, we do not have any finite time periods $\Delta T_i - T_i$, during which there are no delayed vehicles and where the departure flow $O_i(t)$ agrees with the arrival flow A_i . Therefore,

$O_i(t) = \gamma_i(t)\widehat{Q}_i$ according to Eq. (12), and certain relationships simplify. For example, the utilization is given as integral of the permeability over one cycle time T_{cyc} , divided by the cycle time itself:

$$u_i = \frac{1}{T_{\text{cyc}}} \int_{t_{i0}}^{t_{i0}+T_{\text{cyc}}} dt' \gamma_i(t'). \quad (42)$$

Moreover, in the case of constant in- and outflows, i.e. linear increase and decrease of the queue length, the average number ΔN_i^{av} of delayed vehicles is just given by half of the maximum number of delayed vehicles:¹

$$\Delta N_i^{\text{av}} = \frac{\Delta N_i^{\text{max}}}{2}. \quad (43)$$

As a consequence, we have

$$\mathcal{T}_i(\{u_j\}) = \frac{L_i}{V_i^0} + \frac{(1-u_i)T_{\text{los}}}{2(1-\sum_j u_j)}. \quad (44)$$

The average speed V_i^{av} of traffic stream i is often determined by dividing the length L_i of the road section reserved for it by the average travel time $\mathcal{T}_i = \mathcal{T}_i^0 + \mathcal{T}_i^{\text{av}}$, which gives

$$V_i^{\text{av}}(\{u_j\}) = \frac{L_i}{\mathcal{T}_i(\{u_j\})} = \left(\frac{1}{V_i^0} + \frac{(1-u_i)T_{\text{los}}}{2L_i(1-\sum_j u_j)} \right)^{-1}, \quad (45)$$

and can be generalized with Eq. (28) to cases with an efficiencies $\epsilon_i \neq 0$:

$$V_i^{\text{av}}(\{u_j\}, \epsilon_i) = \left(\frac{1}{V_i^0} + (1-\epsilon_i) \frac{(1-u_i)T_{\text{los}}}{2L_i(1-\sum_j u_j)} \right)^{-1}. \quad (46)$$

Note, however, that the above formulas for the average speed are implicitly based on a harmonic rather than an arithmetic average. When correcting for this, Eq. (45), for example, becomes

$$V_i^{\text{av}}(\{u_j\}) = \frac{L_i u_i \widehat{Q}_i}{\Delta N_i^{\text{max}}(\{u_j\})} \ln \left| 1 + \frac{\Delta N_i^{\text{max}}(\{u_j\})}{u_i \widehat{Q}_i T_i^0} \right|. \quad (47)$$

As is shown in Appendix A, this has a similar Taylor approximation as the harmonic average (46). The latter is therefore reasonable to use, and it is simpler to calculate.

As expected from queuing theory, the average travel time (44) diverges, when the sum of utilizations reaches

¹ Note that the formulas derived in this paper are exact under the assumption of continuous flows. The fact that vehicle flows consist of discrete vehicles implies deviations from our formulas of upto 1 vehicle, which slightly affects the average travel times as well. In Fig. 2, for example, the number of vehicles in the queue is $N_i^{\text{max}} - 1$ rather than N_i^{max} . Therefore, the related maximum delay time is $(1-u_i)T_{\text{cyc}}(N_i^{\text{max}} - 1)/N_i^{\text{max}}$, which reduces the average delay time T_i^{av} by $(1-u_i)T_{\text{cyc}}/(2N_i^{\text{max}})$.

the intersection capacity, i.e. $\sum_j u_j \rightarrow 1$. In this practically relevant case, the traffic light would not switch anymore, which would frustrate drivers. For this reason, the cycle time is limited to a finite value

$$T_{\text{cyc}}^{\text{max}}(\{u_j^0\}) = \frac{T_{\text{los}}}{1 - \sum_j u_j^0}, \quad (48)$$

where typically $u_j^0 \leq u_j$. This implies that the sum of utilizations must fulfill

$$\sum_j u_j \leq \sum_j u_j^0 = 1 - \frac{T_{\text{los}}}{T_{\text{cyc}}^{\text{max}}}. \quad (49)$$

As soon as this condition is violated, we will have an increase of the average number of delayed vehicles in time, which characterizes the congested regime discussed in the next section.

4 Fundamental Relationships for Congested Traffic Conditions

In the congested regime, the number of delayed vehicles does not reach zero anymore, and platoons cannot be served without delay. Vehicles will usually have to wait several cycle times until they can finally pass the traffic light. This increases the average delay time enormously. It also implies that there are no excess green times, which means $\delta_i = 0$. Consequently, we can also assume $O_i(t) = \widehat{Q}_i$, as long as the outflow from road sections during the green phase is not (yet) obstructed (otherwise see Sec. 5). Formula (13) applies again and implies for time-independent arrival flows A_i

$$\begin{aligned} \Delta N_i(t_{i0} + kT_{\text{cyc}}^{\text{max}}) &= \Delta N_i(t_{i0}) + \int_{t_{i0}}^{t_{i0} + kT_{\text{cyc}}^{\text{max}}} dt' [A_i - \gamma_i(t')\widehat{Q}_i] \\ &= \Delta N_i(t_{i0}) + (A_i - u_i^0 \widehat{Q}_i)kT_{\text{cyc}}^{\text{max}}, \end{aligned} \quad (50)$$

where

$$u_i^0 = \frac{1}{kT_{\text{cyc}}^{\text{max}}} \int_{t_{i0}}^{t_{i0} + kT_{\text{cyc}}^{\text{max}}} dt' \gamma_i(t') < u_i = \frac{A_i}{\widehat{Q}_i} \quad (51)$$

is the green time fraction of the cycle time $T_{\text{cyc}}^{\text{max}}$ reserved for vehicle stream i . Therefore, the number of delayed vehicles grows by an amount $(A_i - u_i^0 \widehat{Q}_i)T_{\text{cyc}}^{\text{max}} = (u_i - u_i^0)\widehat{Q}_i kT_{\text{cyc}}^{\text{max}}$ in each cycle time $T_{\text{cyc}}^{\text{max}}$. The minimum number during one cycle is

$$\Delta N_i^{\text{min}}(u_i, k) = \Delta N_i(t_{i0}) + (u_i - u_i^0)\widehat{Q}_i kT_{\text{cyc}}^{\text{max}}, \quad (52)$$

and considering Eq. (15), the maximum number of delayed vehicles is

$$\Delta N_i^{\text{max}}(u_i, k) = \Delta N_i^{\text{min}}(u_i, k) + u_i(1 - u_i^0)\widehat{Q}_i kT_{\text{cyc}}^{\text{max}}. \quad (53)$$

Because of $\Delta N_i^{\text{av}} = (\Delta N_i^{\text{min}} + \Delta N_i^{\text{max}})/2$ and $A_i = u_i \widehat{Q}_i$, the average number of delayed vehicles is

$$\Delta N_i^{\text{av}}(u_i, \{u_j^0\}, k) = \left((u_i - u_i^0)k + \frac{u_i(1 - u_i^0)}{2} \right) \widehat{Q}_i T_{\text{cyc}}^{\text{max}}(\{u_j^0\}). \quad (54)$$

Obviously, the average density

$$\rho_i^{\text{av}}(u_i, \{u_j^0\}, L_i, k) = \left((u_i - u_i^0)k + \frac{u_i(1 - u_i^0)}{2} \right) \frac{\widehat{Q}_i T_{\text{cyc}}^{\text{max}}(\{u_j^0\})}{L_i} \quad (55)$$

is obtained by dividing the previous formula by L_i .

It seems logical to use Eq. (54) to determine the average delay time as

$$\begin{aligned} \mathcal{T}_i^{\text{av}}(u_i, \{u_j^0\}, k) &= \frac{\Delta N_i^{\text{av}}(u_i, \{u_j^0\}, k)}{A_i} \\ &= \left(1 - \frac{u_i^0}{u_i} \right) k T_{\text{cyc}}^{\text{max}}(\{u_j^0\}) \\ &\quad + \frac{(1 - u_i^0) T_{\text{cyc}}^{\text{max}}(\{u_j^0\})}{2}. \end{aligned} \quad (56)$$

However, this turns out to be *not* correct, as some of this time is actually not lost, but used to move forward (see Fig. 3). After this first simple analysis, we need to look at the problem more carefully, and perform the calculation without the use of Eq. (24), as Little's Law is not applicable in case of non-stable, systematically growing queues.

Again, we apply the fact that the number of vehicles arriving in one cycle is $A_i T_{\text{cyc}}^{\text{max}}$, while the number of vehicles served during one green time period of duration $u_i^0 T_{\text{cyc}}^{\text{max}}$ is given by $\widehat{Q}_i u_i^0 T_{\text{cyc}}^{\text{max}}$. The difference of both numbers is added to the growing vehicle queue. The quotient of the number $k A_i T_{\text{cyc}}^{\text{max}}$ of vehicles arriving in k cycles and the number $\widehat{Q}_i u_i^0 T_{\text{cyc}}^{\text{max}}$ of vehicles served during one green time period, when rounded down, corresponds to the number n_s of additional stops needed by newly arriving vehicles. Therefore, with $A_i = u_i \widehat{Q}_i$ we have

$$n_s(u_i, \{u_j^0\}, k) = \left\lfloor \frac{A_i k T_{\text{cyc}}^{\text{max}}(\{u_j^0\})}{\widehat{Q}_i u_i^0 T_{\text{cyc}}^{\text{max}}(\{u_j^0\})} \right\rfloor = \left\lfloor \frac{u_i k}{u_i^0} \right\rfloor. \quad (57)$$

Here, $\lfloor x \rfloor$ represents the floor function rounding to lower integers (i.e. cutting the digits after the period, if $x > 0$). Assuming that t_{i0} is the time at which congestion sets in, Eq. (57) can be generalized to continuous time t , allowing us to estimate the number of additional stops of a vehicle arriving at time t :

$$n_s(u_i, \{u_j^0\}, t) = \left\lfloor \frac{A_i(t - t_{i0})}{\widehat{Q}_i u_i^0 T_{\text{cyc}}^{\text{max}}(\{u_j^0\})} \right\rfloor = \left\lfloor \frac{u_i(t - t_{i0})}{u_i^0 T_{\text{cyc}}^{\text{max}}(\{u_j^0\})} \right\rfloor. \quad (58)$$

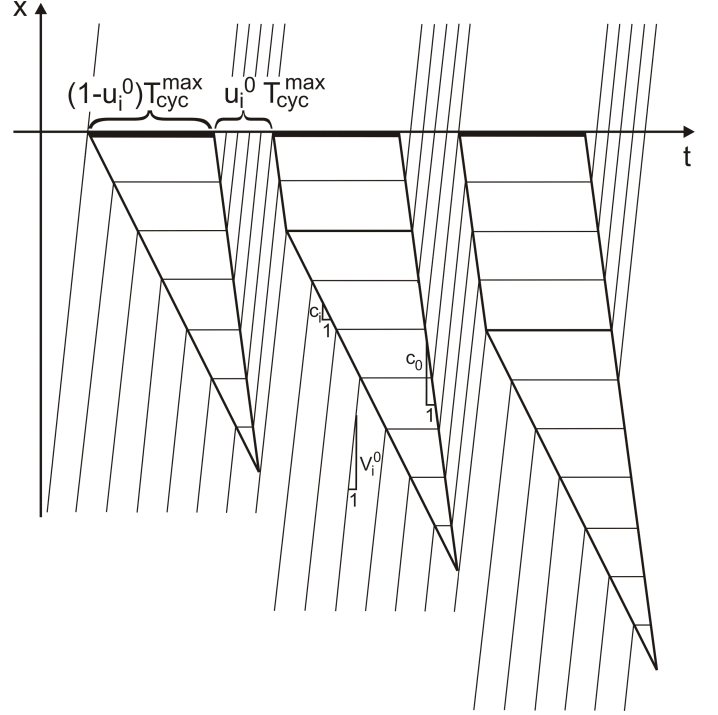


Fig. 3. Illustration of growing vehicle queues in the congested regime. Since not all vehicles can be served within one green time $u_i^0 T_{\text{cyc}}^{\text{max}}$, vehicles are forced to make additional stops, which correspond to the rhomboidal parts of the queue in the above time-space diagram. The triangular part, in contrast, represents newly arriving vehicles.

As Fig. 3 shows, the delay time by each of the n_s additional stops is $(1 - u_i^0) T_{\text{cyc}}^{\text{max}}$. Moreover, we can see that the triangular part of the vehicle queue in the space-time plot gives a further contribution to the delay time of vehicles. Applying Eq. (20) with $f_i = u_i = u_i^0$, the average time delay in this triangular part is $(1 - u_i^0) T_{\text{cyc}}^{\text{max}}/2$, i.e. the arithmetic average between zero and the sum of the red and amber time period (amounting to $(1 - u_i^0) T_{\text{cyc}}^{\text{max}}$). In summary, the delay time of a newly arriving vehicle at time t (when averaging over the triangular part for the sake of simplicity), is

$$\begin{aligned} \mathcal{T}_i^{\text{av}}(u_i, t) &= (1 - u_i^0) \frac{T_{\text{cyc}}^{\text{max}}}{2} + \left\lfloor \frac{u_i(t - t_{i0})}{u_i^0 T_{\text{cyc}}^{\text{max}}} \right\rfloor (1 - u_i^0) T_{\text{cyc}}^{\text{max}} \\ &= \left(\frac{1}{2} + \left\lfloor \frac{u_i(t - t_{i0})}{u_i^0 T_{\text{cyc}}^{\text{max}}} \right\rfloor \right) (1 - u_i^0) T_{\text{cyc}}^{\text{max}}. \end{aligned} \quad (59)$$

Accordingly, the average travel time does not only grow with time t (or the number k of cycles passed), it also grows *stepwise* due to the floor function $\lfloor x \rfloor$.

When we also average formula (59) over its steps, we obtain the approximate relationship

$$\begin{aligned} \mathcal{T}_i^{\text{av}}(u_i, t) &\approx \left(\frac{u_i(t - t_{i0})}{u_i^0 T_{\text{cyc}}^{\text{max}}} \right) (1 - u_i^0) T_{\text{cyc}}^{\text{max}} \\ &= u_i(t - t_{i0}) \frac{(1 - u_i^0)}{u_i^0}, \end{aligned} \quad (60)$$

where it is important to consider that the floor function $\lfloor x \rfloor$ is shifted by 0.5 with respect to the function $\text{round}(x)$, which rounds to the closest integer: $\text{round}(x) = \lfloor x + 0.5 \rfloor$. Moreover, when averaging over the last $(k + 1)$ st cycle we get

$$T_i^{\text{av}}(u_i, k) \approx u_i \left(k + \frac{1}{2} \right) \frac{1 - u_i^0}{u_i^0} T_{\text{cyc}}^{\text{max}}. \quad (61)$$

It is also possible to replace the dependence on the number k of cycles by a dependence on the average density of delayed vehicles by applying Eq. (54). In this way, we obtain

$$k + \frac{1}{2} = \frac{\Delta N_i^{\text{av}}(u_i, \{u_j^0\}, k)}{(u_i - u_i^0) \widehat{Q}_i T_{\text{cyc}}^{\text{max}}(\{u_j^0\})} - \frac{u_i^0(1 - u_i)}{2(u_i - u_i^0)}. \quad (62)$$

Therefore, while in Sec. 3 we could express the average travel time and the average velocity either in dependence of the average density ρ_i^{av} or the utilization u_i alone, we now have a dependence on both quantities.

Finally note that Eqs. (59) to (61) may be generalized to the case where the arrival rate of vehicles is not time-independent. This changes the triangular part of Fig. 3. In order to reflect this, the corresponding contribution $(1 - u_i^0)T_{\text{cyc}}^{\text{max}}/2$ may, again, be multiplied with a prefactor $(1 - \epsilon_i)$, which defines an efficiency ϵ_i . While $\epsilon_i = 0$ corresponds to the previously discussed case of an uniform arrival of vehicles, $\epsilon_i = 1$ reflects the case where a densely packed platoon of vehicles arrives at the moment when the last vehicle in the queue has started to move forward.

5 Fundamental Relationships for Oversaturated Traffic Conditions

We have seen that, under congested conditions, the number of delayed vehicles is growing on average. Hence, the vehicle queue will eventually fill the road section reserved for vehicle stream i completely. Its maximum storage capacity per lane for delayed vehicles is

$$\Delta N_i^{\text{jam}}(L_i) = L_i \rho_i^{\text{jam}}, \quad (63)$$

where ρ_i^{jam} denotes the maximum density of vehicles per lane. The road section becomes completely congested at the time

$$t = t_{i0} + k T_{\text{cyc}}^{\text{max}} + \Delta t, \quad (64)$$

when $\Delta N_i^{\text{min}}(u_i, k) + A_i \Delta t$ according to Eq. (52) reaches the value ΔN_i^{jam} , which implies

$$\Delta t = \frac{\Delta N_i^{\text{jam}} - \Delta N_i^{\text{min}}(u_i, k)}{A_i}, \quad (65)$$

where

$$k = \left\lfloor \frac{\Delta N_i^{\text{jam}}}{(u_i - u_i^0) T_{\text{cyc}}^{\text{max}}} \right\rfloor, \quad (66)$$

see Eq. (52). The number $n_s + 1$ of stops is given by Eq. (58). See Fig. 4 for an illustration.

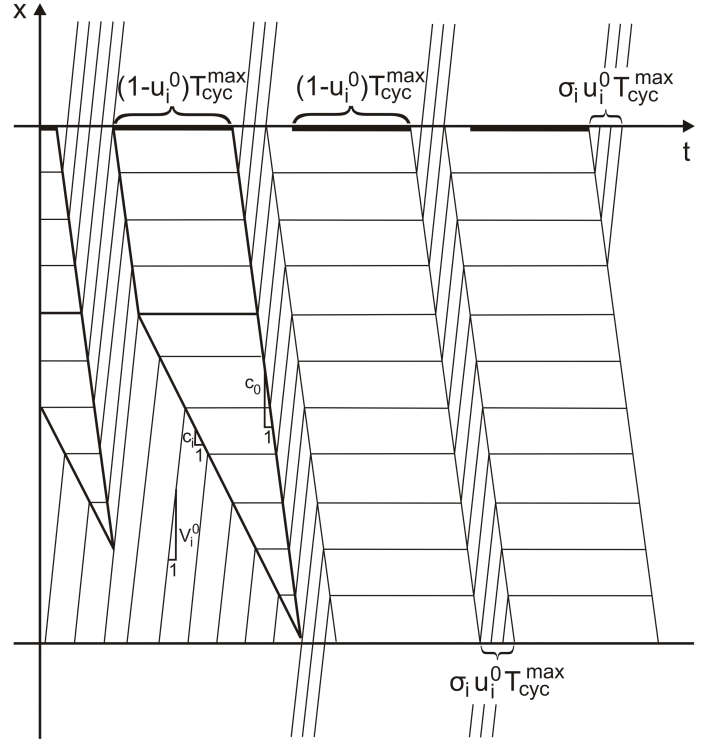


Fig. 4. Illustration of the service of vehicle queues, when the road section is fully congested. The lower horizontal line indicates the location of the upstream end of the road section. It can be seen that vehicles are stopped several times, and that new vehicles can only enter when some vehicles have been served by the traffic light located at the t -axis, and the space freed up by this (which propagates at the characteristic speed c_0) has reached the end of the vehicle queue.

Complete congestion causes spillover effects and obstructs the arrival of upstream vehicles, even though the green phase would, in principle, allow them to depart. When these spill-over effects set in, we are in the oversaturated regime, and only a certain fraction σ_i of the green phase of duration $u_i^0 T_{\text{cyc}}^{\text{max}}$ can be used, where $0 \leq \sigma_i \leq 1$. In this case, $f_i = u_i^0$ must essentially be replaced by $f_i = \sigma_i u_i^0$ in the fundamental relationships for congested traffic, and the lost service time becomes

$$T_{\text{los}} = \sum_i \left[\tau_i + (1 - \sigma_i) u_i^0 T_{\text{cyc}}^{\text{max}} \right]. \quad (67)$$

That is, the reduced service time may be imagined like an extension of the amber time periods. Therefore, it would make sense to *reduce* the cycle time to a value $T_{\text{cyc}} < T_{\text{cyc}}^{\text{max}}$ in the oversaturated regime.

Note, however, that the travel times on the road section reserved for stream i are not growing anymore, because the road section is limited to a length L_i . This allows us to determine the corresponding travel time on the link as follows: The number of vehicles served per cycle time is $\sigma_i u_i^0 T_{\text{cyc}}^{\text{max}} \widehat{Q}_i$. For this reason, the average travel time can

be estimated as

$$\mathcal{T}_i(u_i, \sigma_i, L_i) = \frac{\Delta N_i^{\text{jam}}(L_i)}{\sigma_i u_i^0 T_{\text{cyc}}^{\text{max}}} T_{\text{cyc}}^{\text{max}} = \frac{L_i \rho_i^{\text{jam}}}{\sigma_i u_i^0 \widehat{Q}_i}, \quad (68)$$

and the average delay time $\mathcal{T}_i^{\text{av}} = \mathcal{T}_i - \mathcal{T}_i^0$ as

$$\mathcal{T}_i^{\text{av}}(u_i, \sigma_i, L_i) = \frac{L_i \rho_i^{\text{jam}}}{\sigma_i u_i^0 \widehat{Q}_i} - \frac{L_i}{V_i^0}. \quad (69)$$

Note that these values are now independent of both, the utilization and the average density, as soon as the latter assumes the value $\rho_i^{\text{av}} = \rho_i^{\text{jam}}$, corresponding to a fully congested road section.

5.1 Transition from Oversaturated to Undersaturated Traffic Conditions

If the arrival flow A_i after the rush hour drops below the value of $\sigma_i u_i^0 T_{\text{cyc}}^{\text{max}}$, the vehicle queue will eventually shrink, and the road section used by vehicle stream i enters from the oversaturated into the congested regime. The formulas for the evolution of the number of delayed vehicles are analogous to Eqs. (52 and (53). The queue length starts with ΔN_i^{jam} and is reduced by $(u_i - \sigma_i u_i^0) \widehat{Q}_i k T_{\text{cyc}} < 0$ in each cycle of length T_{cyc} . Counting the number of cycles since the re-entering into the congested regime by k' , we have

$$\Delta N_i^{\text{min}}(u_i, k') = \Delta N_i^{\text{jam}} + (u_i - \sigma_i u_i^0) \widehat{Q}_i k' T_{\text{cyc}}, \quad (70)$$

and considering Eq. (15), the maximum number of delayed vehicles is

$$\Delta N_i^{\text{max}}(u_i, k') = \Delta N_i^{\text{min}}(u_i, k') + u_i (1 - \sigma_i u_i^0) \widehat{Q}_i T_{\text{cyc}}. \quad (71)$$

As soon as $\Delta N_i^{\text{min}}(u_i, k')$ reaches zero, the road section used by vehicle stream i enters the undersaturated regime. Before, the number of stops of vehicles joining the end of the vehicle queue are expected to experience an number $n_s + 1$ of stops with

$$n_s(k') = \left\lfloor \frac{u_i k'}{\sigma_i u_i^0} \right\rfloor, \quad (72)$$

compare Eq. (57).

6 Summary and Outlook

Based on a few elementary assumptions, we were able to derive fundamental relationships for the average travel time $\mathcal{T}_i^{\text{av}}$ and average velocity V_i^{av} . These relationships are functions of the utilization u_i of the service capacity of a cyclically signalized intersection and/or the average number ΔN_i^{av} of delayed vehicles (or the average density ρ_i^{av} of vehicles in the road section of length L_i reserved for traffic stream i). We found different formulas, (1) for the undersaturated regime, (2) for the congested regime,

and (3) for the oversaturated regime. While we also discussed situations, where fixed cycle times T_{cyc}^0 are applied, we primarily focussed on situations, where the cycle time is adjusted to the utilization u_i (in the undersaturated regime) and to the effectively usable green time fraction σ_i (in the oversaturated regime). Our results are summarized in Fig. 5), where also a comparison with the capacity restraint function (1) is made.

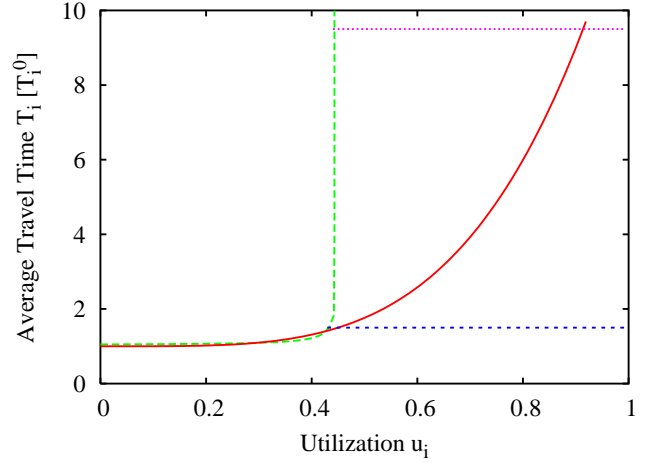


Fig. 5. Illustration of the capacity restraint function (1) that the Bureau of Public Roads recommends to use [16] (solid line), together with analytical results of this paper (dashed lines). Travel time \mathcal{T}_i is measured in units of \mathcal{T}_i^0 . The green long-dashed line shows that the travel time diverges at $u_i \approx 0.45$, if two green phases during one cycle and identical utilizations $u_1 = u_2$ of both associated road sections are assumed, furthermore, if the parameters are set to $T_{\text{los}} = 0.1 \mathcal{T}_i^0$ and $\delta = 0.1$. (A generalization to traffic operation with more than two green phases is easily possible.) When the cycle time is limited to a finite value $T_{\text{cyc}}^{\text{max}}$ to avoid infinite delay times in one of the vehicle queues, one will have growing vehicle queues and increasing travel times in both road sections. Therefore, the travel time can assume any value above the lower dashed horizontal line. The link travel time is only limited by the above dashed horizontal line, which corresponds to the situation where the vehicle queue fills the road section completely. Note that the capacity restraint function (solid line) averages over all travel time measurements for a given utilization u_i . In the right part of the diagram this concerns measurements that depend on the duration of congestion and scatter between both horizontal dashed lines. Here, the curve is shown for $\alpha_i = 0.5$, $\beta_i = 4$, and $A_i/C_i = u_i/0.45$.

The formulas for the *non-congested regime* can be either expressed as non-trivial functions of the utilization u_i or the average queue length ΔN_i^{av} (or the average density ρ_i^{max}). They contain a fit parameter ϵ_i , which reflects effects of variations in the arrival flow and relates to the efficiency of traffic signal operation in terms of synchronizing with vehicle platoons. In the best case, delay times

are zero, which shows the great optimization potential for traffic control in this regime.

In the *congested regime*, the number of delayed vehicles grows in time, and the majority of vehicles is stopped several times by the same traffic light. Therefore, the average travel time does not only depend on the utilization u_i , but also on the average vehicle queue ΔN_i^{av} (or the average density ρ_i^{av}). Although the traffic light control can still improve the average travel times by synchronizing with the arrival of vehicles, the related efficiency effect is rather limited.

In the *oversaturated regime*, the storage capacity of the road section is fully occupied by delayed vehicles, which obstructs the arrival flow. Therefore, the average travel time of a road section reaches a constant maximum value. Nevertheless, the travel time of *vehicles* increases further in time due to spillover effects, which trigger the spreading of congestion to upstream road sections. The actually usable fraction of the green time period is described by a parameter σ_i . Synchronization can still reach some improvements. The most favorable control is oriented at a fluent upstream propagation of the little remaining free space (the difference between ΔN_i^{max} and ΔN_i^{min}). That is, rather than minimizing the delay of downstream moving vehicle platoons, one should now minimize the delay in filling upstream moving gaps [40]. Furthermore, note that the free travel time $T_i^0 = L_i/V_i^0$ and the delay time in the oversaturated regime are proportional to the length L_i of the road section, while the delay times in the undersaturated and congested regimes are independent of L_i .

Based on the above results, it is obvious that it cannot be very successful to describe the travel time of a link by a capacity restraint function which depends on $A_i/C_i = u_i/u_i^0$ only. This basically means an averaging over data that, in principle, are also dependent on the average queue length ΔN_i^{av} (or on the time passed since the onset of congestion). It is, therefore, no wonder that empirical data of travel times as a function of the utilization scatter so enormously in the congested and oversaturated regimes (see e.g. Ref. [27]), that a fitting of the data to functional dependencies of any kind does not make much sense.

This has serious implications for transport modeling, as capacity restraint functions such as formula (1) are used for modeling route choice and, hence, for traffic assignment. Based on the necessary revision of this formula and comparable ones, all traffic scenarios based on these formulas should be critically questioned. Given the computer power of today, it would not constitute a problem to perform a dynamic traffic assignment and routing based on the more differentiated formulas presented in this paper.

We may finally ask ourselves, whether the above formulas would also allow one to make predictions about the average travel times and speeds for a whole *area* of an urban traffic network, rather than for single road sections (“links”) only. In fact, the traveltime-adaptive route choices of drivers could result in a more or less uniform capacity utilization $u_i \approx u$ in a whole area. This would allow one to apply the formulas (28), (59), and (69) to determine the average delay times, and apply them to uniformly uti-

lized areas, considering the average number of traffic lights to be passed. The average velocity would again be given by

$$V^{\text{av}} \approx \frac{\text{average travel distance}}{\text{free travel time} + \text{delay time}}. \quad (73)$$

Empirical data [29] and simulation data [30] provide convincing evidence that averaging over the streets of an urban area works well, and that it provides very useful results.

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A Determination of Average Travel Times and Velocities

Let $f(x)$ be a function and $w(x)$ a weight function. Then, the average of the function between $x = x_0$ and $x = x_1$ is defined as

$$\frac{\int_{x_0}^{x_1} dx' w(x) f(x)}{\int_{x_0}^{x_1} dx' w(x)}. \quad (74)$$

In case of uniform arrivals of vehicles, we have a functional relationship of the form $f(x) = a + bx$ for the travel time, and the weight function is constant, i.e. $w(x) = w$. Here, $a = T_i^0$, $x = \Delta N_i$, and $b = 1/A_i = 1/(u_i \hat{Q}_i)$. With $x_0 = 0$ and $x_1 = \Delta N_i^{\max}$, the formula for the average travel time becomes

$$\frac{w \cdot [(ax_1 + bx_1^2/2) - (ax_0 + bx_0^2/2)]}{w \cdot (x_1 - x_0)} = a + b \frac{x_1 + x_0}{2}, \quad (75)$$

where we have used $(x_1^2 - x_0^2) = (x_1 - x_0)(x_1 + x_0)$. Inserting the above parameters, we obtain the previously derived result

$$T_i = T_i^0 + \frac{\Delta N_i^{\max}}{2u_i \hat{Q}_i}. \quad (76)$$

When determining the average velocity V_i^{av} , the function to average over is of the form $f(x) = c/(a + bx)$, where $c = L_i$ and the other parameters are as defined before. We use the relationship

$$\begin{aligned} \int_{x_0}^{x_1} dx' \frac{wc}{a + bx} &= \frac{wc}{b} \left(\ln |a + bx_1| - \ln |a + bx_0| \right) \\ &= \frac{wc}{b} \ln \left| \frac{a + bx_1}{a + bx_0} \right| \end{aligned} \quad (77)$$

Dividing this again by the normalization factor $w \cdot (x_1 - x_0)$ and inserting the above parameters finally gives

$$V_i^{\text{av}} = \frac{L_i u_i \hat{Q}_i}{\Delta N_i^{\max}} \ln \left| 1 + \frac{\Delta N_i^{\max}}{u_i \hat{Q}_i T_i^0} \right| \approx \frac{L_i}{T_i^0} \left(1 - \frac{\Delta N_i^{\max}}{2u_i \hat{Q}_i T_i^0} \right), \quad (78)$$

where we have used $\ln(1 + x) \leq x - x^2/2$. This formula corrects the naive formula

$$V_i^{\text{av}} \approx \frac{L_i}{T_i} = \frac{L_i}{T_i^0 + \frac{\Delta N_i^{\max}}{2u_i \hat{Q}_i}} \approx \frac{L_i}{T_i^0} \left(1 - \frac{\Delta N_i^{\max}}{2u_i \hat{Q}_i T_i^0} \right), \quad (79)$$

where we have used $1/(1 + x) \approx 1 - x$. Therefore, the above Taylor approximations of both formulas agree, but

higher-order approximations would differ. The formulas in the main part of the paper result for $N_i^{\text{av}} = N_i^{\text{max}}/2$, which corresponds to the case $\delta_i = 0$. However, the above approach can also be generalized to the case $\delta_i > 0$, but in view of the similarity of the Taylor expansions (78) and (79), it does not seem to be worth the effort.