Self-Organized Traffic Flow and Human Coordination in Space and Time

Chair of Sociology, in particular of Modeling and Simulation

Exercises, Sheet 3

---

1. From the common representation of complex numbers

$$z = |z|\mathrm{e}^{\pm\mathrm{i}\varphi} = |z|\big[\cos(\varphi) \pm \mathrm{i}\sin(\varphi)\big]\,, \tag{1}$$

derive the "magic equation"

$$\sqrt{\Re \pm \mathrm{i}|\Im|} = \sqrt{\frac{1}{2}\Big(\sqrt{\Re^2+\Im^2}+\Re\Big)} \pm \mathrm{i}\sqrt{\frac{1}{2}\Big(\sqrt{\Re^2+\Im^2}-\Re\Big)} \tag{2}$$

using only the well-known trigonometric relationship $\sin^2(x)+\cos^2(x) = 1$ and the law $\mathrm{e}^{x_1} \cdot \mathrm{e}^{x_2} = \mathrm{e}^{x_1+x_2}$ which holds for exponential functions. Use the circumstance that the real and imaginary parts of a complex number are orthogonal to each other.

2. Linearize the Lighthill-Whitham-Richards equation and perform the linear stability analysis. What is the real part of the complex eigenvalue of the corresponding characteristic polynomial? What are the implications for the change of the amplitude of the density deviation $\delta\rho(x,t)$ in time? What can one say about the stability of the Burgers equation

$$\frac{\partial\rho(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} = \mathcal{D}\frac{\partial^2\rho(x,t)}{\partial x^2}\,, \tag{3}$$

containing an additional diffusion term $\mathcal{D}\partial^2\rho(x,t)/\partial x^2$?

3. Linearize the continuity equation and the velocity equation

$$\begin{aligned}\frac{\partial V(x,t)}{\partial t} + V(x,t)\frac{\partial V(x,t)}{\partial x} &= -\frac{1}{\rho(x,t)}\frac{\partial \mathcal{P}(x,t)}{\partial x} \\ &+ \frac{1}{\tau}\big[V^{\mathrm{e}}(\rho,V) - V(x,t)\big]\,,\end{aligned} \tag{4}$$

in the case where the pressure $\mathcal{P}(\rho,V)$ and the dynamic equilibrium velocity $V^{\mathrm{e}}(\rho,V)$ are not only dependent on the density $\rho(x,t)$, but also on the average velocity $V(x,t)$. (For simplicity, we have dropped the viscosity term, as this does not lead to qualitatively different findings.) Perform the linear stability analysis and determine the condition for unstable traffic flows. Does the speed-dependence of the traffic pressure and the dynamic equilibrium velocity increase or decrease the stability of traffic flows? How does the instability condition look like for Payne's macroscopic traffic model where

$$\mathcal{P}(\rho) = \frac{V^0 - V_e(\rho)}{2\tau}.$$

4. Compare the instability condition for the optimal velocity model with the one of Payne's macroscopic traffic model. What is the interpretation of the result?

5. Assuming a Gaussian wave packet

$$\delta\rho_0^l(\kappa) = \frac{\mathrm{e}^{-(\kappa-\kappa_0)^2/(2\theta)}}{\sqrt{2\pi\theta}} \tag{5}$$

   and a given dependency $\omega_l(\kappa)$ between wave number κ and frequency ω_l, show that the superposition behaves like a wave with frequency $\omega_l(\kappa_0)$ and speed $\omega_l(\kappa_0)/\kappa_0$, but its amplitude is moving with the group velocity $C_l = d\omega_l(\kappa)/d\kappa$. What relationship exists between the variance of wave numbers and the variance of the amplitude in space?