A Mathematical Model for Attitude Formation by Pair Interactions

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Abstract

Two complementary mathematical models for attitude formation are considered: Starting from the model of WEIDLICH and HAAG (1983), which assumes indirect interactions that are mediated by a mean field, a new model is proposed, which is characterized by direct pair interactions. Three types of pair interactions leading to attitude changes can be found: First, changes by some kind of avoidance behavior. Second, changes by a readiness for compromises. Third, changes by persuasion. Different types of behavior are distinguished by introducing several subpopulations. Representative solutions of the model are illustrated by computational results.

KEY WORDS: attitude formation, direct interactions, indirect interactions, mean field, master equation, probability distribution of attitudes, persuasion, avoidance behavior, compromises, oscillatory attitude changes, frequency locking, interacting populations.
TYPE OF ARTICLE: mathematical model and theory.
DIMENSIONS AND UNITS: transition probabilities per unit time (transition rates); and others.

1 Introduction and Summary

In the field of attitude research there is a broad and substantial literature available (see e.g. FISHBEIN & AJZEN (1975), PETTY & CACIOPPO (1986) for an overview). Especially, some *formal* models have been developed like OSGOOD and TANNENBAUM's (1955) *congruity principle*, HEIDER's (1946) *balance theory* or FESTINGER's (1957) *dissonance theory*. These models deal with the *stability* of attitude structures and the probabilities for special types of of attitude changes.

In contrast, this paper shall treat general mathematical models for the change of the *fraction* of individuals having a certain attitude i. Such models are of great interest for a *quantitative* understanding of attitude formation, and, especially, for the *prognosis* of trends in the public opinion (e.g. of tendencies in the behavior of voters or consumers). COLEMAN's (1964) *diffusion model* was one of the first quantitative models of this kind. It already distinguished two different types of attitude changes: First, changes by direct interactions. Second, externally induced changes. Later, BARTHOLOMEW (1967) developed a *stochastic* model for the *spread of rumors.*

The most advanced model has been introduced by WEIDLICH (WEIDLICH (1972), WEIDLICH & HAAG (1983)), which has been seized by some authors in one or another modification (e.g. TROITZSCH (1989), SCHWEITZER et. al. (1991)). It is based on a *master equation* (a stochastic formulation), which assumes attitude changes to occur with a certain *probability* per unit time (sect. 2). Whereas WEIDLICH and HAAG (1983) assumed attitude changes by *indirect interactions* (which could only describe attitude changes via the media or the spirit of age) (sect. 2), the author thinks that attitude changes by *direct interactions* (due to conversations or discussions) are of special interest. Therefore, a modified master equation is introduced in section 3, which allows to examine the effects of *pair interactions* (dyadic interactions). From this master equation, certain *rate equations* for the temporal development of the *attitude distribution* can be derived (sect. 3).

Three types of of attitude changes by pair interactions can be found (sect. 3): First, an *avoidance behavior*, causing an individual to change the attitude if meeting another individual with the same attitude (defiant behavior, snob effect). Second, a readiness for *compromises.* Third, the tendency to take over the opinion of another individual by *persuasion.* All three types of attitude changes may, in the course of time, lead to stable equilibrium fractions of the attitudes, which depend on the *preferences* for them. However, there may also appear more complex attitude changes which could not be predicted solely by *qualitative* analyses, for example, *oscillatory* or *chaotic* attitude changes.

In section 4, possible solutions of the model are examined by considering representative cases. By distinguishing several *subpopulations* (e.g. blue and white collars), different types of behavior are taken into account. The mutual influence of the subpopulations is assumed to depend on their mutual sympathy.

2 Indirect interactions

In the following we shall shortly discuss the modell of WEIDLICH and HAAG. For a more detailled description see WEIDLICH (1972) and WEIDLICH & HAAG (1983).

Suppose we have a system consisting of N individuals. These individuals can normally be divided into A *subpopulations* a consisting of N_a individuals, i.e.

$$\sum_{a=1}^{A} N_a = N \, . \tag{1}$$

By subpopulations a, different social groups (e.g. blue and white collars) or different characteristic *types* a of behavior are distinguished with respect to the situation of interest (e.g. the voting behavior). The N_a individuals of each subpopulation a are distributed over several *states*

$$\vec{x} \in \{\vec{x}_1, \dots, \vec{x}_{S_a}\} \, . \tag{2}$$

Being in state

$$\vec{x} := (\vec{r}, i) \tag{3}$$

means being at place $\vec{r}$ and having the attitude i concerning the question of interest (e.g. "Which political party i would you vote for, if there were elections today?"). If the *occupation number* $n^a_{\vec{x}}$ denotes the number of individuals of type a being in state $\vec{x}$, we have the relation

$$\sum_{\vec{x}} n^a_{\vec{x}} \equiv \sum_{s=1}^{S_a} n^a_{\vec{x}_s} = N_a\,. \tag{4}$$

Let

$$\vec{n} := (n^1_{\vec{x}_1}, \dots, n^1_{\vec{x}_{S_1}}; \dots; n^A_{\vec{x}_1}, \dots, n^A_{\vec{x}_{S_A}}) = (\vec{n}^1; \dots; \vec{n}^A) \tag{5}$$

be the vector consisting of all occupation numbers $n^a_{\vec{x}}$. This vector is called the *socio-configuration*, since it contains all information about the distribution of the N individuals over the states $\vec{x}_s$. $P(\vec{n}, t)$ shall denote the *probability* to find the socio-configuration $\vec{n}$ at time t. This implies

$$0 \le P(\vec{n}, t) \le 1 \qquad \text{and} \qquad \sum_{\vec{n}} P(\vec{n}, t) = 1\,. \tag{6}$$

If transitions from socio-configuration $\vec{n}$ to $\vec{n}'$ occur with a probability of $P(\vec{n}', t+\Delta t|\vec{n}, t)$ during a short time interval Δt, we have a *(relative) transition rate* of

$$w(\vec{n}', \vec{n}; t) = \lim_{\Delta t \to 0} \frac{P(\vec{n}', t + \Delta t|\vec{n}, t)}{\Delta t}\,. \tag{7}$$

The *absolute* transition rate of changes from $\vec{n}$ to $\vec{n}'$ is the product $w(\vec{n}', \vec{n}; t)P(\vec{n}, t)$ of the probability $P(\vec{n}, t)$ to have the configuration $\vec{n}$ and the *relative* transition rate $w(\vec{n}', \vec{n}; t)$ if having the configuration $\vec{n}$. Whereas the *inflow* into $\vec{n}$ is given as the sum over all absolute transition rates of changes from an *arbitrary* configuration $\vec{n}'$ to $\vec{n}$, the *outflow* from $\vec{n}$ is given as the sum over all absolute transition rates of changes from $\vec{n}$ to *another* configuration $\vec{n}'$. Since the temporal change of the probability $P(\vec{n}, t)$ is determined by the inflow into $\vec{n}$ reduced by the outflow from $\vec{n}$, we find the *master equation*

$$\begin{aligned} \frac{d}{dt}P(\vec{n}, t) &= \text{inflow into } \vec{n} \qquad\qquad - \text{ outflow from } \vec{n} \\ &= \sum_{\vec{n}'} w(\vec{n}, \vec{n}'; t)P(\vec{n}', t) - \sum_{\vec{n}'} w(\vec{n}', \vec{n}; t)P(\vec{n}, t) \end{aligned} \tag{8}$$

(HAKEN (1983)).

WEIDLICH and HAAG (1983) have assumed the individuals to change from state $\vec{x}$ to state $\vec{x}'$ with a transition rate of $w_a(\vec{x}', \vec{x}; \vec{n}; t)$ *independently* of each other. Such changes correspond to transitions of the socio-configuration from $\vec{n}$ to

$$\vec{n}^{aa}_{\vec{y}\vec{x}} := (n^1_{\vec{x}_1}, \dots, n^1_{\vec{x}_{S_1}}; \dots; n^a_{\vec{x}_1}, \dots, (n^a_{\vec{y}}+1), \dots, (n^a_{\vec{x}}-1), \dots, n^a_{\vec{x}_{S_a}}; \dots; n^A_{\vec{x}_1}, \dots, n^A_{\vec{x}_{S_A}}) \tag{9}$$

with a transition rate of $n^a_{\vec{x}} w_a(\vec{x}', \vec{x}; \vec{n}; t)$, which is proportional to the number $n^a_{\vec{x}}$ of individuals who can leave the state $\vec{x}$. For $w(\vec{n}', \vec{n}; t)$ we therefore have the relation

$$w(\vec{n}', \vec{n}; t) = \begin{cases} n^a_{\vec{x}} w_a(\vec{x}', \vec{x}; \vec{n}; t) & \text{if } \vec{n}' = \vec{n}^{aa}_{\vec{y}\vec{x}} \\ 0 & \text{else}\,. \end{cases} \tag{10}$$

Assumption (10) has sometimes been doubted to be suitable for describing social processes, since it handles the interactions of individuals in an *indirect* way via the dependence on the occupation numbers $n^a_{\vec{x}}$. That means, the individual interactions are mediated by the *mean field* of the socio-configuration, which could model attitude changes via the media or the spirit of the age. Figure 1 illustrates the $2N$ indirect interactions of N individuals.

In contrast, this article will deal with *direct* interactions of individuals, which are a realistic model for attitude changes in conversations or discussions. We shall consider *pair interactions* only, because they are the most important ones. However, it is also possible to develop models for simultaneous interactions of an arbitrary number of individuals (HELBING (1991a)). Figure 2 depicts the $N(N-1)$ direct pair interactions of N individuals.

3 Direct pair interactions

A model describing direct pair interactions is given by the master equation (8), when taking the special transition rates

$$w(\vec{n}',\vec{n};t) = \begin{cases} n^a_{\vec{x}} n^b_{\vec{y}} \widetilde{w}_{ab}(\vec{x}',\vec{y}';\vec{x},\vec{y};t) & \text{if } \vec{n}' = \vec{n}^{a\,b\,ab}_{\vec{x}'\vec{y}'\vec{x}\vec{y}} \\ 0 & \text{else}, \end{cases} \tag{11}$$

where

$$\vec{n}^{a\,b\,ab}_{\vec{x}'\vec{y}'\vec{x}\vec{y}} := (\dots,(n^a_{\vec{x}'}+1),\dots,(n^a_{\vec{x}}-1),\dots,(n^b_{\vec{y}'}+1),\dots,(n^b_{\vec{y}}-1),\dots)\,. \tag{12}$$

$\widetilde{w}_{ab}(\vec{x}',\vec{y}';\vec{x},\vec{y};t)$ is the (relative) transition rate for two individuals of types a and b to change their states from $\vec{x}$ and $\vec{y}$ to $\vec{x}'$ and $\vec{y}'$ due to direct interactions. The corresponding transition rate $n^a_{\vec{x}} n^b_{\vec{y}} \widetilde{w}_{ab}(\vec{x}',\vec{y}';\vec{x},\vec{y};t)$ for changes of the configuration from $\vec{n}$ to $\vec{n}^{a\,b\,ab}_{\vec{x}'\vec{y}'\vec{x}\vec{y}}$ is again proportional to the numbers $n^a_{\vec{x}}$ and $n^b_{\vec{y}}$ of individuals which can leave the states $\vec{x}$ and $\vec{y}$.

Usually, one is mostly interested in the expected *fractions*

$$P_a(\vec{x},t) := \frac{\langle n^a_{\vec{x}} \rangle}{N_a} \tag{13}$$

of individuals of type a who are in state $\vec{x}$. Here, $\langle n^a_{\vec{x}} \rangle$ denotes the mean value

$$\langle n^a_{\vec{x}} \rangle := \sum_{\vec{n}} n^a_{\vec{x}} P(\vec{n},t) \tag{14}$$

of the occupation number $n^a_{\vec{x}}$. For $P_a(\vec{x},t)$ we have the relations

$$0 \le P_a(\vec{x},t) \le 1 \qquad \text{and} \qquad \sum_{\vec{x}} P_a(\vec{x},t) = 1\,. \tag{15}$$

$P_a(\vec{x},t)$ can be interpreted as the probability distribution of state $\vec{x}$ within subpopulation a. Since N_a/N is the fraction of individuals of type a, the probability distribution $P(\vec{x},t)$ of state $\vec{x}$ in the *entire* population is given by

$$P(\vec{x},t) = \sum_a \frac{N_a}{N} P_a(\vec{x},t)\,. \tag{16}$$

The temporal development of $P_a(\vec{x},t)$ obeys the BOLTZMANN equation (BOLTZMANN (1964))

$$\frac{d}{dt}P_a(\vec{x},t) = \sum_b \sum_{\vec{x}'} \sum_{\vec{y}} \sum_{\vec{y}'} w_{ab}(\vec{x},\vec{y}';\vec{x}',\vec{y};t) P_a(\vec{x}',t) P_b(\vec{y},t) \qquad (17a)$$

$$- \sum_b \sum_{\vec{x}'} \sum_{\vec{y}} \sum_{\vec{y}'} w_{ab}(\vec{x}',\vec{y}';\vec{x},\vec{y};t) P_a(\vec{x},t) P_b(\vec{y},t) \qquad (17b)$$

with

$$w_{ab} := N_b \widetilde{w}_{ab} \,, \qquad (18)$$

as long as the absolute values of the (co)variances

$$\sigma^{ab}_{\vec{x}\vec{x}'} := \sum_{\vec{n}} (n^a_{\vec{x}} - \langle n^a_{\vec{x}} \rangle)(n^b_{\vec{x}'} - \langle n^b_{\vec{x}'} \rangle) P(\vec{n},t) = \langle n^a_{\vec{x}} n^b_{\vec{x}'} \rangle - \langle n^a_{\vec{x}} \rangle \langle n^b_{\vec{x}'} \rangle \qquad (19)$$

are small compared to $\langle n^a_{\vec{x}} \rangle \langle n^b_{\vec{x}'} \rangle$, and if $(N_a - 1)/N_a \approx 1$ (see HELBING (1991a)). These conditions are usually fulfilled in case of a great population size N. Equation (17) can be also understood as *exact* equation for the *most probable* individual behavior.

The interpretation of (17) is similar to the one of the master equation (8). Again, the temporal change of the fraction $P_a(\vec{x},t)$ of individuals being in state $\vec{x}$ is given by the inflow (17a) into state $\vec{x}$ reduced by the outflow (17b) from state $\vec{x}$:

$$\frac{d}{dt}P_a(\vec{x},t) = \text{inflow into } \vec{x} - \text{outflow from } \vec{x} \,. \qquad (20)$$

The inflow [resp. outflow] is given by all transitions, where individuals of type a change their states from arbitrary states $\vec{x}'$ to state $\vec{x}$ [resp. from state $\vec{x}$ to arbitrary states $\vec{x}'$] due to pair interactions with other individuals of arbitrary type b, who change their arbitrary states $\vec{y}$ to arbitrary states $\vec{y}'$. These pair interactions occur with a frequency proportional to the fractions P_a and P_b of the states which are subject to a change. Since b, $\vec{x}'$, $\vec{y}$ and $\vec{y}'$ are arbitrary, one has to carry out a summation over them.

In the following we shall consider the cases of *spatially homogeneous* or *local* attitude formation, which are independent of $\vec{r}$. Equation (17) has then the form

$$\frac{d}{dt}P_a(i,t) = \sum_b \sum_{i'} \sum_j \Big[w^*_{ab}(i;i',j;t) P_a(i',t) P_b(j,t) - w^*_{ab}(i';i,j;t) P_a(i,t) P_b(j,t) \Big] \,, \qquad (21)$$

where the abbreviation

$$w^*_{ab}(.;.,.;t) := \sum_{j'} w_{ab}(.,j';.,.;t) \qquad (22)$$

has been used. $w^*_{ab}(i';i,j;t)$ is the rate for individuals of type a to change from attitude i to attitude i' due to interactions with individuals of type b having the attitude j. For the description or prediction of concrete situations the rates w^*_{ab} have to be determined empirically (see HELBING (1991b)). In the following, we shall instead examine the possible solutions of (21) by considering representative examples. Only three kinds of interaction contribute to the temporal change of $P_a(i,t)$:

$$i' \stackrel{i}{\longleftarrow} i \quad (i' \neq i) \tag{23.1}$$
$$i' \stackrel{j}{\longleftarrow} i \quad (j \neq i,\, i' \neq i,\, i' \neq j) \tag{23.2}$$
$$j \stackrel{j}{\longleftarrow} i \quad (j \neq i) \tag{23.3}$$

The interpretation is obviously the following:

1. The first term describes some kind of *avoidance behavior,* causing an individual to change the opinion from i to i' with a certain probability, when meeting another individual originally having the same opinion i. Such defiant behavior is e.g. known as snob effect.

2. The second term represents the readiness to change the opinion from i to $i' \neq j$ (in the following called a *compromise*), if meeting another individual having a different opinion $j \neq i$. This behavior will be found, if attitude i cannot be maintained when confronted with attitude j, but attitude j is not satisfying (for this person) either.

3. The third term describes the tendency to take over the opinion j of another individual by *persuasion.*

According to this classification, the transition rates $w^*_{ab}(i';i,j)$ can be splitted into three contributions:

$$\begin{aligned} w^*_{ab}(i';i,j;t) &= w^{*1}_{ab}(i';i;t)\delta_{ij} && (24.1)\\ &+ w^{*2}_{ab}(i';i,j;t) && (24.2)\\ &+ w^{*3}_{ab}(j;i;t)\delta_{ji'}\,, && (24.3)\end{aligned}$$

where

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{25}$$

is the KRONECKER function and

$$w^{*1}_{ab}(i,i) = 0, \quad w^{*2}_{ab}(i;i,j) = 0, \quad w^{*2}_{ab}(j;i,j) = 0, \quad w^{*2}_{ab}(i';i,i) = 0, \quad w^{*3}_{ab}(i;i) = 0\,. \tag{26}$$

Equation (21) has now the form

$$\begin{aligned} \frac{d}{dt}P_a(i,t) &= \sum_b \sum_{i'} \Big[w^{*1}_{ab}(i;i';t)P_a(i',t)P_b(i',t) - w^{*1}_{ab}(i';i;t)P_a(i,t)P_b(i,t) \Big] && (27.1)\\ &+ \sum_b \sum_{i'} \sum_j \Big[w^{*2}_{ab}(i;i',j;t)P_a(i',t)P_b(j,t) - w^{*2}_{ab}(i';i,j;t)P_a(i,t)P_b(j,t) \Big] && (27.2)\\ &+ \sum_b \sum_j \Big[w^{*3}_{ab}(i;j;t)P_a(j,t)P_b(i,t) - w^{*3}_{ab}(j;i;t)P_a(i,t)P_b(j,t) \Big]\,. && (27.3)\end{aligned}$$

w^{*k}_{ab} $(k = 1,2,3)$ are the contributions to the transition rates w^*_{ab} by interactions of type k.

4 Computer solutions

In order to obtain concrete results, we have to make some plausible specifications of the model. Let κ_{ab} be the degree of *sympathy* which individuals of type a feel towards individuals of type b. Then one expects the following: Whereas the rate w^{*2}_{ab} of the readiness for compromises and the rate w^{*3}_{ab} of the tendency to take over another opinion will be increasing with κ_{ab}, the rate w^{*1}_{ab} of the avoidance behavior will be decreasing with κ_{ab}. This functional dependence can e.g. be described by

$$
\begin{aligned}
w^{*1}_{ab}(i';i;t) &:= \nu^1_{ab}(t)R_a(i',i;t)(1-\delta_{ii'}) && (28.1)\\
w^{*2}_{ab}(i';i,j;t) &:= \nu^2_{ab}(t)R_a(i',i;t)(1-\delta_{ii'})(1-\delta_{ij})(1-\delta_{ji'}) && (28.2)\\
w^{*3}_{ab}(j;i;t) &:= \nu^3_{ab}(t)R_a(j,i;t)(1-\delta_{ij}) && (28.3)
\end{aligned}
$$

with

$$
\begin{aligned}
\nu^1_{ab}(t) &:= \nu^1_a(1-\kappa_{ab}) && (29.1)\\
\nu^2_{ab}(t) &:= \nu^2_a\kappa_{ab} && (29.2)\\
\nu^3_{ab}(t) &:= \nu^3_a\kappa_{ab} && (29.3)
\end{aligned}
$$

($0 \le \kappa_{ab} \le 1$). The factors $(1-\delta_{ij})$ are due to (26). ν^k_a is a measure for the rate of attitude changes of type k within subpopulation a, i.e. a *flexibility parameter*. The rates $\nu^k_{ab}(t)$ can be understood as product of the *contact rate* $\nu_{ab}(t)$ of an individual of subpopulation a with individuals of subpopulation b, and the frequency $f^k_{ab}(t)$ of having an interaction of type k in case of a contact:

$$\nu^k_{ab}(t) = \nu_{ab}(t) f^k_{ab}(t)\,. \qquad (30)$$

ν^k_{ab} has the effect of a *coupling coefficient* and determines, how strong attitude changes of kind k within subpopulation a depend on the attitude distribution $P_b(j,t)$ within subpopulation b. Especially $\nu^k_{ab} = 0$ for $k = 1,2,3$ implies, that the dynamic development of $P_a(i,t)$ is completely independent of $P_b(j,t)$ (see fig. 3).

For the case of two subpopulations we explicitly have the sympathy matrix

$$\underline{\kappa} \equiv \Big(\kappa_{ab}\Big) = \begin{pmatrix} 1 & \kappa_{12} \\ \kappa_{21} & 1 \end{pmatrix}, \qquad (31)$$

if the sympathy between individuals belonging to the same subpopulation is assumed to be maximal ($\kappa_{11} = 1 = \kappa_{22}$). Then, the abbreviation

$$\underline{\kappa}^c_d := \begin{pmatrix} 1 & c \\ d & 1 \end{pmatrix} \qquad (32)$$

can be used. Figures 3 to 5 ($\nu^3_1 = (S_1-1) = 3$, $\nu^3_2 = (S_2-1) = 2$, $\nu^1_a = \nu^2_a = 0$, $\underline{R}_1 = \underline{R}^4$, $\underline{R}_2 = \underline{R}^3$) demonstrate the effects of variations of $\underline{\kappa}^c_d$. Attitudes of subpopulation $a = 1$ are represented by solid lines, attitudes of subpolulation $a = 2$ by broken lines. Figure 3

($\underline{\kappa} = \underline{\kappa}_0^0$) is simulated *without* any coupling of the subpopulations, figure 4 ($\underline{\kappa} = \underline{\kappa}_1^0$) with *asymmetric* coupling and figure 5 ($\underline{\kappa} = \underline{\kappa}_1^1$) with *mutual* coupling.

$R_a(i', i; t)$ is a measure for the *readiness to change the attitude* from i to i' for an individual of type a, if an attitude change takes place. We shall assume

$$R_a(i, i'; t) = \frac{1}{S_a - 1} - R_a(i', i; t) \quad \text{with} \quad 0 \leq R_a(i', i; t) \leq \frac{1}{S_a - 1}, \quad \text{if } i \neq i' \tag{33}$$

(S_a = number of different attitudes within subpopulation a). That means, the readiness to change from opinion i' to i will be the greater, the lower the readiness for the inverse change from i to i' is.

$$P_{a,i}(t) := \sum_{i'(\neq i)} R_a(i, i'; t) \tag{34}$$

can be interpreted as degree of *preference* for opinion i (see figures 6 and 7). Since $R_a(i, i; t)$ is arbitrary (see (28), (33)), we may define

$$R_a(i, i; t) := P_{a,i}(t) \, . \tag{35}$$

Usually, the equilibrium fraction will be growing with the preference $P_{a,i}$: see figure 6 ($\nu_a^k = S_a - 1 = 2$, $\underline{\kappa} = \underline{\kappa}_1^0$, $\underline{R}_1 = \underline{R}^1$, $\underline{R}_2 = \underline{R}^2$), where a solid line represents the highest preference, a dotted line the lowest preference, and a broken line medium preference. For the special case

$$P_{a,i} = (S_a - 1)\frac{1}{2(S_a - 1)} = \frac{1}{2}, \tag{36}$$

one of the stationary solutions is

$$P_a(i) = \frac{1}{S_a} \, . \tag{37}$$

This is because of

$$\sum_{i'(\neq i)} R_a(i', i) = \sum_{i'(\neq i)} \left(\frac{1}{S_a - 1} - R_a(i, i') \right) = 1 - P_{a,i} = \frac{1}{2} = P_{a,i} = \sum_{i'(\neq i)} R_a(i, i') \tag{38}$$

and equations (27) to (29). An illustration of this case is given in figure 7 ($\nu_a^k = S_a - 1 = 2$, $\underline{\kappa} = \underline{\kappa}_1^0$, $\underline{R}_a = \underline{R}^3$).

In the following, we shall often consider the situation of three different opinions ($S_a = 3$):

$$\underline{R}_a \equiv \Big(R_a(i', i) \Big) := \begin{pmatrix} P_{a,1} & \frac{1}{S_a - 1} - A_a & \frac{1}{S_a - 1} - B_a \\ A_a & P_{a,2} & \frac{1}{S_a - 1} - C_a \\ B_a & C_a & P_{a,3} \end{pmatrix} . \tag{39}$$

(A_a is the readiness to change from attitude 1 to attitude 2, B_a to change from 1 to 3, and C_a to change from 2 to 3.) Especially, for

$$\underline{R}_a = \underline{R}^1 := \begin{pmatrix} 0.7 & 0.35 & 0.35 \\ 0.15 & 0.5 & 0.35 \\ 0.15 & 0.15 & 0.3 \end{pmatrix} , \tag{40}$$

attitude 1 has the highest preference ($P_{a,1} = 0.7$), attitude 3 has the lowest preference ($P_{a,3} = 0.3$), and attitude 2 has medium preference ($P_{a,2} = 0.5$). For

$$\underline{R}_a = \underline{R}^2 := \begin{pmatrix} 0.5 & 0.15 & 0.35 \\ 0.35 & 0.7 & 0.35 \\ 0.15 & 0.15 & 0.3 \end{pmatrix} , \tag{41}$$

attitude 3 has again the lowest preference ($P_{a,3} = 0.3$), but attitude 2 has the highest preference ($P_{a,2} = 0.7$), and attitude 1 has medium preference ($P_{a,1} = 0.5$). If

$$\underline{R}_a = \underline{R}^3 := \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} , \tag{42}$$

all attitudes have the same preference ($P_{a,i} = 1/2$). An analogous case for 4 attitudes is given by

$$\underline{R}_a = \underline{R}^4 := \begin{pmatrix} 1/2 & 0 & 1/6 & 1/3 \\ 1/3 & 1/2 & 0 & 1/6 \\ 1/6 & 1/3 & 1/2 & 0 \\ 0 & 1/6 & 1/3 & 1/2 \end{pmatrix} . \tag{43}$$

However, the case of three different opinions represents the prototype of opinion formation, since the attitudes concerning a special question can be classified within the schemes "positive", "negative", "neutral" or "pro", "contra", "compromise".

4.1 Avoidance behavior

In this section we shall examine the effect of avoidance behavior alone (i.e. $\nu_a^1 = S_a - 1 = 2$, $\nu_a^2 = \nu_a^3 = 0$). Figures 8 and 9 compare the case, where both subpopulations prefer attitude 1 (Fig. 8: $\underline{\kappa} = \underline{\kappa}_{0.5}^{0.5}$, $\underline{R}_a = \underline{R}^1$) and where they prefer different attitudes (Fig. 9: $\underline{\kappa} = \underline{\kappa}_{0.5}^{0.5}$, $\underline{R}_1 = \underline{R}^1$, $\underline{R}_2 = \underline{R}^2$). The attitude i with the highest preference is represented by a solid line, the attitude with the lowest preference by a dotted line, and the one with medium preference by a broken line. Obviously, the fraction of the most prefered attitude is decreasing, if both subpopulations favour the same attitude (see fig. 8). If they prefer different attitudes, it is growing (see fig. 9), since there are less situations of avoidance then.

Avoidance behavior is e.g. known as snob effect. It also occurs in the case of hostile groups.

4.2 Readiness for compromises

Let us now consider the effect of persuasion in combination with a readiness for compromises (see figures 10 and 11: $\nu_a^1 = 0$, $\nu_a^2 = \nu_a^3 = S_a - 1 = 2$). For the effect of persuasion *without* readiness for compromises (i.e. $\nu_a^1 = \nu_a^2 = 0$, $\nu_a^3 = S_a - 1 = 2$) see the corresponding figures 12 and 13. Again, we shall compare the case, where both subpopulations prefer attitude 1 (Fig. 10: $\underline{\kappa} = \underline{\kappa}_1^1$, $\underline{R}_a = \underline{R}^1$), with the case, where they prefer different attitudes (Fig. 11: $\underline{\kappa} = \underline{\kappa}_1^1$, $\underline{R}_1 = \underline{R}^1$, $\underline{R}_2 = \underline{R}^2$). If both subpopulations prefer the same attitude, this attitude will be the only surviving one (see fig. 10). The readiness for compromises will have little influence then (compare to fig. 12: $\underline{\kappa} = \underline{\kappa}_1^1$, $\underline{R}_a = \underline{R}^1$). However, if the subpopulations favour different attitudes, the compromise (the 3rd attitude, represented by the dotted line) will be chosen by a certain fraction of individuals (see fig. 11). According to this, the competition between several attitudes will lead to a greater variety of attitudes, if there is a readiness for compromises. Without this readiness ($\nu_a^2 = 0$), the competing attitudes will be the only surviving ones (see fig. 13: $\underline{\kappa} = \underline{\kappa}_1^1$, $\underline{R}_1 = \underline{R}^1$, $\underline{R}_2 = \underline{R}^2$). However, they will survive in *both* subpopulations (at least, if w_{ab}^{*3} is time independent).

4.3 Effects of persuasion

This section will deal with the tendency to take over the opinion j of another individual (i.e. $\nu_a^3 = S_a - 1$, $\nu_a^1 = \nu_a^2 = 0$). Typical cases are depicted in figures 12 and 13. However, there are also *oscillations* possible as shown in figures 3, 4 and 14 ($\underline{\kappa} = \underline{\kappa}_1^1$, $\underline{R}_a = \underline{R}^3$). In order to understand this behavior, let us consider the case of a single subpopulation ($A = 1$) or of uncoupled subpopulations ($\kappa_{ab} = 0$ for all a, b with $a \neq b$). From equations (27c), (28c), (29c) we then obtain

$$\frac{d}{dt} P_a(i,t) = P_a(i,t) \sum_j M_{ij}^a(t) P_a(j,t) \tag{44}$$

with

$$\underline{M}^a(t) \equiv \Big(M_{ij}^a(t) \Big) := \Big(\nu_a^3 \kappa_{aa} \big[R_a(i,j;t) - R_a(j,i;t) \big] \Big) . \tag{45}$$

Equation (44) has at least the S_a stationary solutions

$$P_a^s(i) := \delta_{is} = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{if } i \neq s \end{cases} , \qquad s = 1, \dots, S_a . \tag{46}$$

A linear stability analysis with

$$\delta P_a^s(i,t) := P_a(i,t) - P_a^s(i) , \qquad \delta P_a^s(i,t) \ll 1 \tag{47}$$

leads to the equations

$$\frac{d}{dt} \delta P_a^s(i,t) = \begin{cases} -M_{si}^a \delta P_a^s(i,t) & \text{for } i \neq s \\ \sum_j M_{ij}^a \delta P_a^s(j,t) & \text{for } i = s \end{cases} \tag{48}$$

(where $\delta P_a^s(i,t) > 0$ for $i \neq s$, $\delta P_a^s(s,t) = -\sum_{i(\neq s)} \delta P_a^s(i,t) < 0$). Therefore, the stationary solution $P_a^s(i)$ is *stable* if

$$M_{si}^a > 0 , \quad \text{i.e.} \quad R_a(s,i) > R_a(i,s) \qquad \text{for every } i \text{ with } i \neq s. \tag{49}$$

Otherwise it is unstable. Because of $M_{si}^a = -M_{is}^a$ (see (45)), there is at most one stationary solution $P_a^l(i)$ stable. Attitude l will be the only surviving attitude then. If none of the solutions $P^s_a(i)$ is stable, there may exist another stationary solution $P^0_a(i)$, which is given by

$$\sum_j M^a_{ij} P^0_a(j) = 0 \quad \text{for every } i. \tag{50}$$

A linear stability analysis with (47) leads to

$$\frac{d}{dt} \delta P^0_a(i,t) = P^0_a(i) \sum_j M^a_{ij} \delta P^0_a(j,t) . \tag{51}$$

Since $\underline{M}^a$ is an antisymmetric matrix, i.e.

$$\left(M^a_{ji} \right) = - \left(M^a_{ij} \right) \tag{52}$$

(see (45)), equation (51) has only imaginary eigenvalues (MAY (1973)). Therefore, an oscillatory behavior of $\delta P^0_a(i,t)$ and $P_a(i,t)$ results. (52) also implies that

$$\sum_i \Big[P_a(i,t) - P^0_a(i) \ln P_a(i,t) \Big] = \text{const.}, \quad \text{i.e.} \quad \prod_i P_a(i,t)^{P^0_a(i)} = \text{const.} \tag{53}$$

(see MAY (1973)). As a consequence of (53) and (15), $P_a(i,t)$ moves on a $(S_a - 2)$-dimensional hypersurface (see figures 15, 18 and 19). Especially for $S_a = 3$ attitudes, we find a *cyclical* movement. This can be seen in figure 15 ($\underline{R}_a = \underline{R}^3$), where simulations for different initial values are depicted.

Let us examine the case with $\underline{R}_a = \underline{R}^3$ in detail. $\underline{M}^a$ is of the form

$$\left(M^a_{ij} \right) = \nu^3_a \kappa_{aa} \begin{pmatrix} 0 & -1/2 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{pmatrix} , \tag{54}$$

then. As a consequence, the stationary solutions $P^s_a(i)$ with $s = 1,2,3$ are unstable (see (49)). $P^0_a(i) \equiv 1/S_a$ is another stationary solution and has imaginary eigenvalues. Equation (44) takes the form

$$\frac{dP_a(i,t)}{dt} = \frac{\nu^3_a \kappa_{aa}}{S_a - 1} P_a(i,t) \Big[P_a(i-1,t) - P_a(i+1,t) \Big] \equiv \lambda^a_i(t) P_a(i,t) \quad \text{with} \quad i \equiv i \bmod S_a . \tag{55}$$

$\lambda^a_i(t) := \nu^3_a \kappa_{aa} [P_a(i-1,t) - P_a(i+1,t)]/(S_a - 1)$ is a *dynamic order parameter* and can be interpreted as time dependent *growth rate*. From (55) one can see the following:

- $P_a(i,t)$ increases as long as $P_a(i-1,t) > P_a(i+1,t)$, i.e. $\lambda^a_i(t) > 0$.
- The growth of $P_a(i,t)$ induces a growth of $P_a(i+1,t)$ by $\lambda^a_{i+1}(t) > 0$, and a decrease of $P_a(i-1,t)$ by $\lambda^a_{i-1}(t) < 0$.
- As soon as $P_a(i+1,t)$ exceeds $P_a(i-1,t)$ the decrease of $P_a(i,t)$ begins, because of $\lambda^a_i(t) < 0$.
- These phases are repeated again and again due to $i \equiv i \bmod S_a$ (i.e. a "cyclical sequence of the S_a attitudes").

The above situation can occur, if the readiness to exchange the present attitude i for the previous attitude $(i-1)$ is less than the readiness to exchange i for a new attitude $(i+1)$. After a finite number S_a of changes one may return to the original attitude, and the process starts from the beginning. Typical examples are the behavior of consumers (concerning fashion) or voters.

In figures 16 and 17 ($A = 1$), the solution of equation (55) for a varying number S_1 of attitudes is depicted. With growing S_1, the temporal development of the fractions $P_1(i,t)$ becomes more complex. For $S_1 = 5$ attitudes, $P_1(i,t)$ looks already quite irregular (see fig. 17). However, a *phase plot*[1] illustrates that $P_1(i,t)$ is still periodic, since it shows a *closed* curve (see fig. 18). A similar situation can be found for $S_a = 6$ attitudes (see fig. 19). For $S_a = 7$ even the phase plot appears irregular.

Let us return to figure 3, where two subpopulations periodically change between $S_1 = 4$ resp. $S_2 = 3$ attitudes with different frequencies independently of each other. The complex ratio between these frequencies can be easily seen in the corresponding phase plot (see fig. 20: $\underline{\kappa} = \underline{\kappa}_0^0$). However, if subpopulation $a = 2$ is influenced by subpopulation $a = 1$ (see fig. 4), the phase plot differs drastically (see fig. 21: $\underline{\kappa} = \underline{\kappa}_1^0$). Obviously, the oscillations of $P_2(i,t)$ in subpopulation 2 have the same frequency as in subpopulation 1, then. This remarkable adaption of frequency is called *frequency locking*.

5 Outlook

We have developed a model for attitude formation by pair interactions, which can be written in the form

$$\frac{d}{dt}P_a(i,t) = \sum_{i'} \Big[w^a(i,i';t)P_a(i',t) - w^a(i',i;t)P_a(i,t) \Big] \qquad (56)$$

with the *effective* transition rates

$$w^a(i',i;t) := R_a(i',i;t) \sum_b \Big[\big(\nu_{ab}^1(t) - \nu_{ab}^2(t)\big) P_b(i,t) + \nu_{ab}^2(t) + \big(\nu_{ab}^3(t) - \nu_{ab}^2(t)\big) P_b(i',t) \Big] \qquad (57)$$

(see equations (27) and (28)). This model includes attitude changes by an avoidance behavior, by a readiness for compromises and by persuasion. It also takes into account different types of behavior by distinguishing several subpopulations a.

The model allows some modifications and generalizations, which shall be discussed in forthcoming pubplications:

- Equations of type (17) can produce *chaotic* attitude changes (HELBING (1991c)), i.e. a temporal behavior that is unpredictable, since it depends on the initial conditions in a very sensible way (SCHUSTER (1984), HAO (1984)).

[1]Here, the term *phase plot* has the meaning of a two-dimensional projektion of the trajectory $(P_1(1,t),\dots,P_A(S_A,t))$. For example, a plot of the value $P_a(2,t)$ over the value $P_a(1,t)$ with varying time t would be a special case of a phase plot.

- Other interesting effects are generated by spatial interactions in attitude formation (HELBING (1991e)). Such spatial interactions may result from migration (SCHWEITZER et. al. (1991), WEIDLICH & HAAG (1988)) or by telecommunication.

- Attitude formation could be reformulated in a way that allows to understand attitude changes as a reaction on a *social field* (HELBING (1991c), LEWIN (1951), SPIEGEL (1961)).

- There are *several* plausible specifications of the readiness $R_a(i',i;t)$ for attitude changes from i to i':

  - In the model of this paper, assumptions (33) and (35) imply the relation

$$\begin{aligned}\sum_{i'} R_a(i',i;t) &= \sum_{i'(\neq i)} \left(\frac{1}{S_a-1} - R_a(i,i';t)\right) + R_a(i,i;t) \\ &= 1 - P_{a,i}(t) + P_{a,i}(t) = 1\,, \end{aligned} \qquad (58)$$

    and $R_a(i',i;t)$ could be interpreted as the *probability* for an individual of subpopulation a to change the attitude from i to i' during a time interval $\tau^a := 1/\nu^a$, where

$$\nu^a(t) := \sum_b \left[\Big(\nu^1_{ab}(t) - \nu^2_{ab}(t)\Big)P_b(i,t) + \nu^2_{ab}(t) + \Big(\nu^3_{ab}(t) - \nu^2_{ab}(t)\Big)P_b(i',t)\right]. \qquad (59)$$

    Obviously, τ^a depends on the rates of pair interactions of types k.

  - Another plausible specification of $R_a(i',i;t)$ is given by

$$R_a(i',i;t) := \frac{\mathrm{e}^{U_a(i',t)-U_a(i,t)}}{D_a(i',i;t)} \qquad (60)$$

    with

$$D_a(i',i;t) = D_a(i,i';t) \quad \text{and} \quad D_a(i',i,t) > 0 \qquad (61)$$

    (compare to WEIDLICH & HAAG (1983)). That means, the readiness $R_a(i',i;t)$ for an individual of type a to change the attitude from i to i' will be the greater, the greater the difference of the *plausibilities (utilities)* U_a of attitudes i' and i is, and the smaller the *incompatibility (distance)* $D_a(i',i;t)$ of attitudes i and i' is.

  The optimal specification of the model (56), (57) can only be found by evaluation of empirical data.

- Therefore, methods have to be developed, which are able to determine the parameters in the models from (complete or incomplete) empirical data (HELBING (1991b)).

- A quantitative model for the readiness $R_a(i',i;t)$ to exchange an attitude i by an attitude i' may be developed, which calculates the value of $R_a(i',i;t)$ as a function of the *relation* of attitudes i and i'. OSGOOD and TANNENBAUM's (1955) congruity principle, HEIDER's (1946) balance theory and FESTINGER's (1957) dissonance theory have already found valuable results on this problem.

- The models treated in this paper can be generalized to the situation illustrated in figure 22, where both pair interactions and indirect (mean field) interactions play an important role. In that case, we have to use the transition rates

$$w(\vec{n}',\vec{n};t) = \begin{cases} n_{\vec{x}}^a \widetilde{w}_a(\vec{x}',\vec{x};\vec{n};t) & \text{if } \vec{n}' = \vec{n}_{\vec{x}'\vec{x}}^{a\ a} \\ n_{\vec{x}}^a n_{\vec{y}}^b \widetilde{w}_{ab}(\vec{x}',\vec{y}';\vec{x},\vec{y};\vec{n};t) & \text{if } \vec{n}' = \vec{n}_{\vec{x}'\vec{y}'\vec{x}\vec{y}}^{a\ b\ ab} \\ 0 & \text{else} \end{cases} \qquad (62)$$

when solving the master equation (8). The corresponding approximate equations for the expected fractions $P_a(\vec{x},t)$ are, if $(N_a - 1)/N_a \approx 1$:

$$\frac{d}{dt}P_a(\vec{x},t) = \sum_{\vec{x}'} \Big[w^a\big(\vec{x},\vec{x}';\langle\vec{n}\rangle;t\big)P_a(\vec{x}',t) - w^a\big(\vec{x}',\vec{x};\langle\vec{n}\rangle;t\big)P_a(\vec{x},t)\Big] \qquad (63)$$

with

$$w^a\big(\vec{x}',\vec{x};\langle\vec{n}\rangle;t\big) := w_a\big(\vec{x}',\vec{x};\langle\vec{n}\rangle;t\big) + \sum_b \sum_{\vec{y}} \sum_{\vec{y}'} w_{ab}\big(\vec{x}',\vec{y}';\vec{x},\vec{y};\langle\vec{n}\rangle;t\big)P_b(\vec{y},t)\,, \qquad (64)$$

$$w_a := \widetilde{w}_a\,, \qquad w_{ab} := N_b \widetilde{w}_{ab} \qquad (65)$$

and

$$\langle n_{\vec{x}}^a \rangle = N_a P_a(\vec{x},t)\,. \qquad (66)$$

Equations (63), (64) are very general and enable a great spectrum of quantitative models for social processes: Special cases of these equations are

  - the *logistic equation* (VERHULST (1845), PEARL (1924), MONTROLL & BADGER (1974)) and the LOTKA-VOLTERRA equation (LOTKA (1920, 1956), VOLTERRA (1931), GOEL, MAITRA & MONTROLL (1971), HALLAM (1986), GOODWIN (1967)) for population dynamics,
  - the FISHER-EIGEN equation (FISHER (1930), EIGEN (1971), FEISTEL & EBELING (1989)) for the evolution of new (behavioral) strategies (HELBING (1991)),
  - the *gravity model* (RAVENSTEIN (1876), ZIPF (1946)) for migration phenomena, and
  - the models of WEIDLICH and HAAG (WEIDLICH & HAAG (1983, 1988), WEIDLICH (1972, 1991)).

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