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MICROSCOPIC FOUNDATION OF STOCHASTIC GAME DYNAMICAL EQUATIONS

I. INTRODUCTION

Since von Neumann and Morgenstern initiated the field of game theory,[1] it has often proved of great value for the quantitative description and understanding of competition and co-operation between individuals. Game theory focusses on two questions: 1. Which is the optimal strategy in a given situation? 2. What is the dynamics of strategy choices in cases of repeatedly interacting individuals? In this connection game dynamical equations[2] find a steadily increasing interest. Although they agree with the replicator equations of evolution theory (cf. Sec. II), they cannot be justified in the same way. Therefore, we will be looking for a foundation of the game dynamical equations which is based on individual actions and decisions (cf. Sec. IV).

In addition, we will formulate a stochastic version of evolutionary game theory (cf. Sec. III). This allows us to investigate the effects of fluctuations on the dynamics of social systems. In order to illustrate the essential ideas, a concrete model for the self-organization of behavioral conventions is presented (cf. Sec. V). We will see that the game dynamical equations describe the average evolution of social systems only for restricted time periods. Therefore, a criterium for their validity will be developed (cf. Sec. VI). Finally, we will present possible extensions to more general behavioral models and discuss the actual meaning of the game dynamical equations (cf. Sec. VII).

II. THE GAME DYNAMICAL EQUATIONS

Let $p_x(t)$, such that

$$0 \leq p_x(t) \leq 1 \qquad \text{and} \qquad \sum_x p_x(t) = 1, \tag{1}$$

denote the *proportion* of individuals pursuing the *behavioral strategy* $x \in S$ at time t. We assume the strategies considered to be mutually exclusive. The set S of strategies may be discrete or continuous, finite or infinite. The only difference will be that sums over x are to be replaced by integrals in cases of continuous sets. By A_{xy} we will denote the possibly time-dependent *payoff* for an individual using strategy x when confronted with an individual pursuing strategy y. Hence, his/her

W. Leinfellner and E. Köhler (eds.), Game Theory, Experience, Rationality, 211–224.

expected success $\langle E_x \rangle_t$ will be given by the weighted mean value

$$\langle E_x \rangle_t = \sum_y A_{xy}\, p_y(t)\,, \tag{2}$$

since p_y is the probability that the interaction partner uses strategy y. In addition, the *average expected success* will be

$$\overline{\langle E \rangle_t} = \sum_x p_x(t) \langle E_x \rangle_t = \sum_x \sum_y p_x(t) A_{xy}\, p_y(t)\,. \tag{3}$$

Assuming that the relative temporal increase $(dp_x/dt)/p_x$ of the proportion p_x of individuals pursuing strategy x is proportional to the difference between the expected success $\langle E_x \rangle_t$ and the average expected success $\overline{\langle E \rangle_t}$, we obtain the *game dynamical equations*

$$\begin{aligned} \frac{dp_x(t)}{dt} &= \nu p_x(t) \left[\langle E_x \rangle_t - \overline{\langle E \rangle_t} \right] \\ &= \nu p_x(t) \Big[\langle E_x \rangle_t - \sum_y p_y(t) \langle E_y \rangle_t \Big]\,, \end{aligned} \tag{4}$$

where the possibly time-dependent proportionality factor ν is a measure for the *interaction rate* with other individuals. According to (4), the proportions of strategies with an above-average success $\langle E_x \rangle_t > \overline{\langle E \rangle_t}$ increase, whereas the other strategies will be diminished. Note, that the proportion of a strategy does not necessarily increase or decrease monotonically. Certain payoffs are associated with an *oscillatory* or even *chaotic* dynamics[3].

Equations (4) are identical with the replicator equations from evolutionary biology. They can be extended to the *selection-mutation equations*

$$\begin{aligned} \frac{dp_x(t)}{dt} &= \nu p_x(t) \Big[\langle E_x \rangle_t - \sum_y p_y(t) \langle E_y \rangle_t \Big] \\ &+ \sum_y \left[p_y(t) w_1(y \to x) - p_x(t) w_1(x \to y) \right]\,. \end{aligned} \tag{5}$$

The terms which agree with (4) describe a selection of superior strategies. The new terms correspond to the effect of mutations, i.e. to *spontaneous* changes from strategy x to other strategies y with possibly time-dependent *transition rates* $w_1(x \to y)$ (last term) and the inverse transitions. They allow to describe *trial and error behavior* or behavioral fluctuations.

III. Stochastic Dynamics: The Master Equation

Let us consider a social system consisting of a constant number

$$N = \sum_x n_x(t) \tag{6}$$

of individuals. Herein, $n_x(t)$ denotes the number of individuals who pursue strategy x at time t. Hence, the time-dependent vector

$$\vec{n} = (n_1, n_2, \dots, n_x, \dots, n_y, \dots) \tag{7}$$

reflects the *strategy distribution* in the social system and is called the *socioconfiguration*. If the individual strategy changes are subject to random fluctuations (e.g. due to trial and error behavior or decisions under uncertainty), we will have a stochastic dynamics. Therefore, given a certain socioconfiguration $\vec{n}_0$ at time t_0, for the occurence of the strategy distribution $\vec{n}$ at a time $t > t_0$ we can only calculate a certain *probability* $P(\vec{n}, t)$. Its temporal change dP/dt is governed by the so-called *master equation*[4]

$$\frac{dP(\vec{n},t)}{dt} = \sum_{\vec{n}\,'} \left[P(\vec{n}\,',t) W(\vec{n}\,' \to \vec{n}) - P(\vec{n},t) W(\vec{n} \to \vec{n}\,') \right] . \tag{8}$$

The sum over $\vec{n}\,'$ extends over all socioconfigurations fulfilling $n_x \in \{0, 1, 2, \dots\}$ and (6).

According to equation (8), an *increase* of the probability $P(\vec{n}, t)$ of having socioconfiguration $\vec{n}$ is caused by transitions from other socioconfigurations $\vec{n}\,'$ to $\vec{n}$. While a *decrease* of $P(\vec{n}, t)$ is related to changes from $\vec{n}$ to other socioconfigurations $\vec{n}\,'$. The corresponding changing rates are proportional to the *configurational transition rates* $W(\vec{n} \to \vec{n}\,')$ of changes to socioconfigurations $\vec{n}\,'$ *given* the socioconfiguration $\vec{n}$ and to the probability $P(\vec{n}, t)$ of *having* socioconfiguration $\vec{n}$ at time t.

The configurational transition rates W have the meaning of transition probabilities per time unit and must be non-negative quantities. Frequently, the individuals can be assumed to change their strategies independently of each other. Then, the configurational transition rates have the form

$$W(\vec{n} \to \vec{n}\,') = \begin{cases} n_x w(x \to y; \vec{n}) & \text{if } \vec{n}\,' = \vec{n}_{xy} \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

i.e. they are proportional to the number n_x of individuals who may change their strategy from x to another strategy y with an *individual transition rate* $w(x \to y; \vec{n}) \geq 0$. In relation (9), the abbreviation

$$\vec{n}_{xy} = (n_1, n_2, \dots, n_x - 1, \dots, n_y + 1, \dots) \tag{10}$$

means the socioconfiguration which results after an individual has changed his/her strategy from x to y.

It can be shown that the master equation has the properties

$$P(\vec{n}, t) \geq 0 \qquad \text{and} \qquad \sum_{\vec{n}} P(\vec{n}, t) = 1 \tag{11}$$

for all times t, if they are fulfilled at some initial time t_0. Therefore, the master equation actually describes the temporal evolution of a probability distribution.

IV. Approximate Mean Value Equations

In order to connect the stochastic model to the game dynamical equations, we must specify the individual transition rates w in a suitable way. Therefore, we derive the mean value equations related to the master equation (8) and compare them to the selection-mutation equations (5).

The proportion p_x is defined as the *mean value*

$$\langle f \rangle_t = \sum_{\vec{n}} f(\vec{n}, t) P(\vec{n}, t) \tag{12}$$

of the number $f(\vec{n}, t) = n_x$ of individuals pursuing strategy x, divided by the total number N of considered individuals:

$$p_x(t) = \frac{\langle n_x \rangle_t}{N} = \frac{1}{N} \sum_{\vec{n}} n_x P(\vec{n}, t) . \tag{13}$$

Taking the time derivative of $\langle n_x \rangle_t$ and inserting the master equation gives

$$\begin{aligned} \frac{d\langle n_x \rangle_t}{dt} &= \sum_{\vec{n}} n_x \big[P(\vec{n}', t) W(\vec{n}' \to \vec{n}) - P(\vec{n}, t) W(\vec{n} \to \vec{n}') \big] \\ &= \sum_{\vec{n}} (n'_x - n_x) W(\vec{n} \to \vec{n}') P(\vec{n}, t) , \end{aligned} \tag{14}$$

where we have interchanged $\vec{n}$ and $\vec{n}'$ in the first term on the right hand side. Taking into account relation (9), we get

$$\begin{aligned} \frac{d\langle n_x \rangle_t}{dt} &= \sum_{\vec{n}_{yx}} n_y w(y \to x; \vec{n}) P(\vec{n}, t) - \sum_{\vec{n}_{xy}} n_x w(x \to y; \vec{n}) P(\vec{n}, t) \\ &= \sum_{y} \big[n_y w(y \to x; \vec{n}) - n_x w(x \to y; \vec{n}) \big] P(\vec{n}, t) . \end{aligned} \tag{15}$$

With (13) this finally leads to the *approximate mean value equations*

$$\frac{dp_x(t)}{dt} = \sum_{y} \big[p_y(t) w(y \to x; \langle \vec{n} \rangle_t) - p_x(t) w(x \to y; \langle \vec{n} \rangle_t) \big] \tag{16}$$

However, these are only exact if the individual transition rates w are independent of the socioconfiguration $\vec{n}$. In any case, they are *approximately* valid as long as the probability distribution $P(\vec{n}, t)$ is narrow, so that the mean value $\langle f(\vec{n}, t) \rangle_t$ of a function $f(\vec{n}, t)$ can be replaced by the function $f(\langle \vec{n} \rangle_t, t)$ of the mean value. This problem will be discussed in detail later on.

Comparing the rate equations (16) with the selection-mutation equations (5), we find a complete correspondence for the case

$$w(y \to x; \vec{n}) = w_1(y \to x) + w_2(y \to x)\, n_x \tag{17}$$

with

$$w_2(y \to x) = \frac{\nu}{N} \max(E_x - E_y, 0) \tag{18}$$

and the *success*

$$E_x = \sum_y A_{xy} \frac{n_y}{N}, \tag{19}$$

since

$$\max(\langle E_x \rangle_t - \langle E_y \rangle_t, 0) - \max(\langle E_y \rangle_t - \langle E_x \rangle_t, 0) = \langle E_x \rangle_t - \langle E_y \rangle_t . \tag{20}$$

Whereas w_1 is again the mutation rate (i.e. the rate of spontaneous transitions), the additional term in (17) describes *imitation processes,* where individuals take over the strategy x of their respective interaction partner. Imitation processes correspond to pair interactions of the form

$$y + x \to x + x . \tag{21}$$

Their frequency is proportional to the number n_x of interaction partners who may convince an individual of strategy x. The proportionality factor w_2 is the *imitation rate*.

Relation (18) is called the *proportional imitation rule* and can be shown to be the best learning rule.[5]

It was discovered in 1992[6] and says that an imitation behavior only takes place if the strategy x of the interaction partner turns out to have a greater success E_x than one's own strategy y. In such cases, the imitation rate is proportional to the difference $(E_x - E_y)$ between the success of the alternative x and the previous strategy y, i.e. strategy changes occur more often the greater the advantage of the new strategy x would be.

All specifications of the type

$$w_2(y \to x) = C + \frac{\nu}{N} [\lambda E_x - (1 - \lambda) E_y] \tag{22}$$

with an arbitrary parameter λ also lead to our game dynamical equations. However, individuals would then, with a certain rate, take over the strategy x of the interaction partner, even if its success E_x is smaller than that of the previously used strategy y. Moreover, if C is not chosen sufficiently large, the individual transition rates $w \geq 0$ can become negative.

In summary, we have found a microscopic foundation of evolutionary game theory which is based on four plausible assumptions: 1. Individuals evaluate the success of a strategy as its average payoff in interactions with other individuals (cf. (19)). 2. They compare the success of their strategy with that of the respective interaction partner, basing on observations or an exchange of experiences. 3. Individuals imitate each others' behavior. 4. In doing so, they apply the proportional imitation rule (18) [or (22)].

V. SELF-ORGANIZATION OF BEHAVIORAL CONVENTIONS

For illustrative reasons, we will now discuss an example which allows us to understand how social conventions emerge. We consider the simple case of two alternative strategies $x \in \{1, 2\}$ and assume them to be equivalent so that the payoff matrix is symmetrical:

$$\left(A_{xy}\right) = \begin{pmatrix} A+B & B \\ B & A+B \end{pmatrix} . \tag{23}$$

If $A > 0$, the additional payoff A reflects the *advantage* of using the same strategy as the respective interaction partner. This situation holds, for example, in cases of network externalities like in the historical rivalry between the video systems VHS and BETA MAX.[7] Finally, the mutation rates are taken to be constant, i.e. $w_1(x \to y) = W_1$.

The resulting game dynamical equations are

$$\frac{dp_x(t)}{dt} = -2\left[p_x(t) - \frac{1}{2}\right]\left\{W_1 + \nu A p_x(t)\left[p_x(t) - 1\right]\right\} . \tag{24}$$

Obviously, they have only *one* stable stationary solution if the (control) parameter

$$\kappa = 1 - \frac{4W_1}{\nu A} \tag{25}$$

is smaller than zero. However, for $\kappa > 0$ equation (24) can be rewritten in the form

$$\frac{dp_x(t)}{dt} = -2\nu A\left[p_x(t) - \frac{1}{2}\right]\left[p_x(t) - \frac{1+\sqrt{\kappa}}{2}\right]\left[p_x(t) - \frac{1-\sqrt{\kappa}}{2}\right] . \tag{26}$$

The stationary solution $p_x = 1/2$ is unstable, then, but we have two new stable stationary solutions $p_x = (1/2 \pm \sqrt{\kappa}/2)$. That is, dependent on the detailed initial condition, one strategy will gain the majority of users although both strategies are completely equivalent. This phenomenon is called *symmetry breaking*. It will be suppressed if the mutation rate W_1 is larger than the advantage effect $\nu A/4$.

The above model allows us to understand how behavioral conventions come about. Examples are the pedestrians' preference for the right-hand side (in Europe), the revolution direction of clock hands, the direction of writing, or the already mentioned triumph of the video system VHS over BETA MAX.

It is very interesting how the above-mentioned symmetry breaking affects the probability distribution $P(\vec{n}, t) = P(n_1, n_2, t) = P(n_1, N - n_1, t)$ of the related stochastic model (cf. Fig. 1[8]). For $\kappa < 0$ the probability distribution is located around $n_1 = N/2 = n_2$ and stays small, so that the approximate mean value equations are applicable. At the so-called *critical point* $\kappa = 0$, a *phase transition* to a qualitatively different system behavior occurs and the probability distribution becomes very broad. As a consequence, the game dynamical equations do not correctly describe the temporal evolution of the mean strategy distribution anymore.

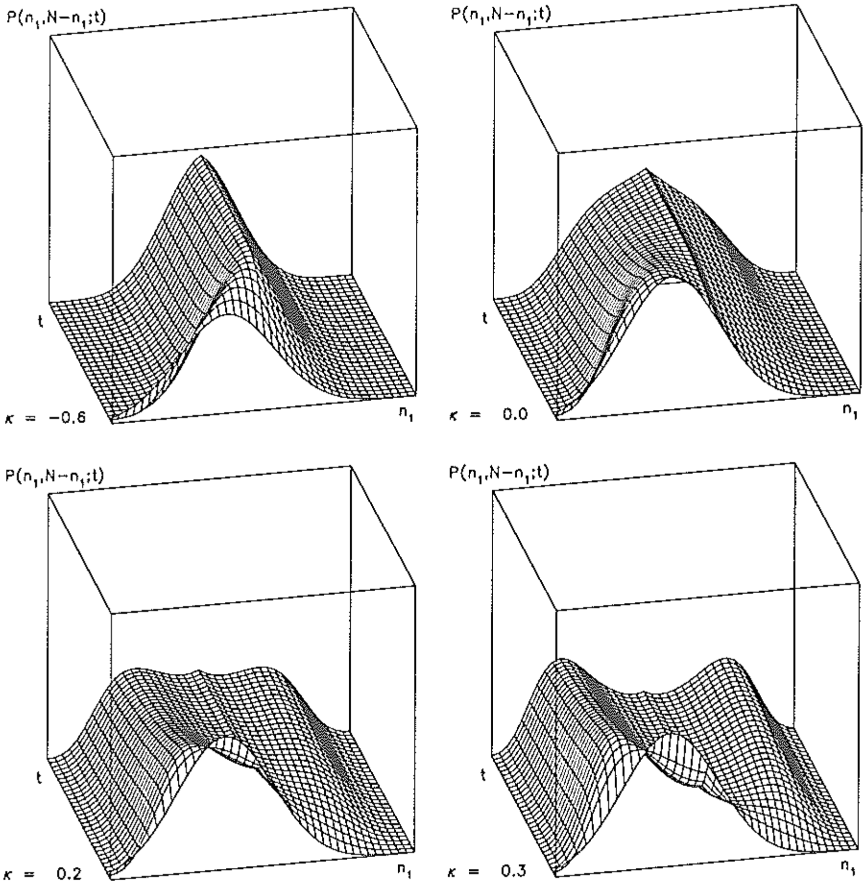

Figure 1: Probability distribution $P(\vec{n}, t) = P(n_1, N - n_1; t)$ of the socioconfiguration $\vec{n}$ for varying values of the control parameter κ according to the stochastic version of the game dynamical equations.

For $\kappa > 0$, a bimodal and symmetrical probability distribution evolves. That is, the likelihood that one of the two equivalent strategies will win throughout is much larger than the likelihood of finding approximately equal proportions of both strategies. At the beginning, the initial state or maybe some random fluctuation determines which strategy has better odds of winning. However, in the long run both strategies have exactly the same odds. It is clear that in such cases the game dynamical equations fail to describe the mean system behavior (cf. Fig. 2) which would correspond to the average temporal evolution of an ensemble of identical social systems. In cases of oscillatory or chaotic solutions of the game dynamical equations the situation is even worse.

VI. Exact, Approximate, and Corrected Mean Values and Variances

In the last section we have seen that the approximate mean value equations

$$\frac{d\langle n_x\rangle_t}{dt} = M_x(\langle\vec{n}\rangle_t), \tag{27}$$

with the so-called *first jump moments*

$$M_x(\vec{n}) = \sum_{\vec{n}\,'}(n'_x - n_x)W(\vec{n} \to \vec{n}\,') \tag{28}$$

(cf. (14)), are not sufficient. This calls for corrected mean value equations and a criterion of validity for the time period. If the individual transition rates $w(x \to y; \vec{n})$ depend on the socioconfiguration, the exact mean value can only be evaluated via formula (13). This requires the calculation of the probability distribution $P(\vec{n}, t)$ and, therefore, the numerical solution of the respective master equation (8). Since the number of possible socioconfigurations is normally very large, an extreme amount of computer time would be necessary for this.

Luckily, it is possible to derive from (14) the *corrected mean value equations*

$$\frac{\partial\langle n_x\rangle_t}{\partial t} = M_x(\langle\vec{n}\rangle_t) + \frac{1}{2}\sum_y\sum_{y'}\sigma_{yy'}(t)\frac{\partial^2 M_x(\langle\vec{n}\rangle_t)}{\partial\langle n_y\rangle_t\partial\langle n_{y'}\rangle_t} \tag{29}$$

by means of a suitable Taylor approximation. This equation depends on the *covariances*

$$\sigma_{xy}(t) = \Big\langle\big(n_x - \langle n_x\rangle_t\big)\big(n_y - \langle n_y\rangle_t\big)\Big\rangle_t = \sum_{\vec{n}}\big(n_x - \langle n_x\rangle_t\big)\big(n_y - \langle n_y\rangle_t\big)P(\vec{n}, t)\,, \tag{30}$$

which can be determined by means of the *covariance equations*

$$\begin{aligned}\frac{\partial\sigma_{xx'}(t)}{\partial t} &= M_{xx'}(\langle\vec{n}\rangle_t) + \frac{1}{2}\sum_y\sum_{y'}\sigma_{yy'}(t)\frac{\partial^2 M_{xx'}(\langle\vec{n}\rangle_t)}{\partial\langle n_y\rangle_t\partial\langle n_{y'}\rangle_t}\\ &+ \sum_y\left[\sigma_{xy}(t)\frac{\partial M_{x'}(\langle\vec{n}\rangle_t)}{\partial\langle n_y\rangle_t} + \sigma_{x'y}(t)\frac{\partial M_x(\langle\vec{n}\rangle_t)}{\partial\langle n_y\rangle_t}\right]. \end{aligned}\tag{31}$$

The functions

$$M_{xy}(\vec{n}) = \sum_{\vec{n}\,'} (n'_x - n_x)(n'_y - n_y) W(\vec{n} \to \vec{n}\,') \tag{32}$$

are called the *second jump moments*.

Equations (29) and (31) build a closed system of equations, but still no exact one, since this would depend on higher moments of the form $\langle n_x n_y n_z \cdots \rangle_t$. Nevertheless, according to Figure 2 the corrected mean value equations yield significantly better results than the approximate ones. As a consequence, they are valid for a much longer time period. Suitable *validity criteria* are the *relative variances*

$$V_x(t) := \frac{\sigma_{xx}(t)}{(\langle n_x \rangle_t)^2}\,, \tag{33}$$

since these are a measure for the relative width of the probability distribution $P(\vec{n}, t)$. It can be shown that the covariances and all higher moments are small, if only $V_x(t)$ is much smaller than 1 for every x. Numerical investigations indicate that the approximate mean value equations begin to separate from the exact ones as soon as one of the relative variances $V_x(t)$ becomes greater than 0.04. The corrected mean value equations and covariances remain reliable as long as $V_x(t)$ is smaller than 0.12 for all x (cf. Fig. 2).

A more detailled discussion of the above matter is presented elsewhere.[9]

VII. DIVERSE GENERALIZATIONS

The behavioral model discussed above can be generalized in different respects.

Modified transition rates: The strange cusp at $n_1 = N/2$ in Figure 1, which comes from the discontinuous derivative of $w_2(x \to y)$ at $E_x = E_y$, can be avoided by using modified imitation rates:

$$w_2(y \to x) = \frac{\nu}{N} \frac{\exp(E_x - E_y)}{D_{xy}} \qquad \text{with} \qquad D_{xy} = D_{yx} = 2\,. \tag{34}$$

This approach agrees with relation (22) in linear approximation for $C = \nu/(2N)$ and $\lambda = 1/2$, but it always yields non-negative imitation rates. As in (18) it guarantees two essential things: 1. The imitation rate grows with an increasing gain $(E_x - E_y)$ of success. 2. If the alternative strategy x is inferior, the imitation rate is very small (but, due to uncertainty, not negligible). The results of the corresponding stochastic behavioral model are presented in Figure 3. They show the usual flatness of the probability distribution $P(n_1, N - n_1, t)$ at the critical point $\kappa = 0$, where again a phase transition occurs.

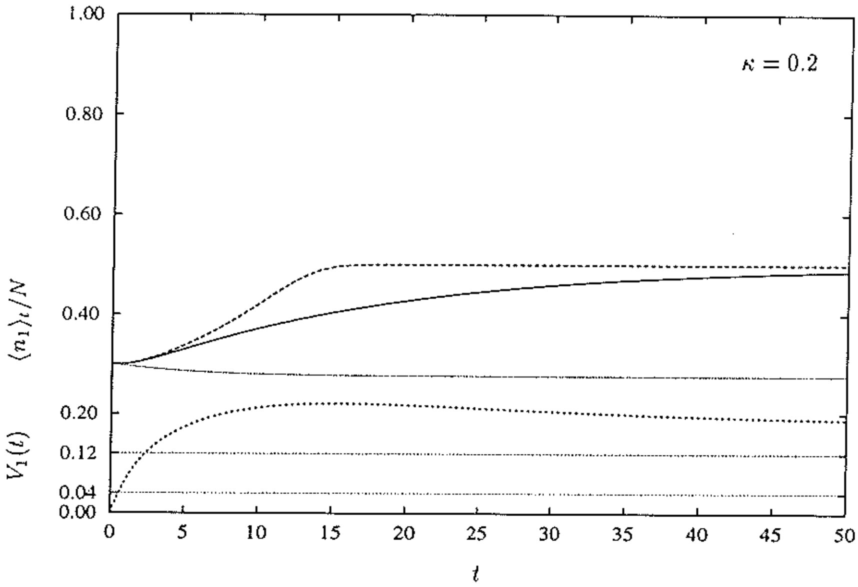

Figure 2: The numerical solutions of the approximate mean value equations (· · ·) agree with those of the exact mean value equations (—) only for a short time interval. The corrected mean value equations (– –) yield much better results, although they also deviate from the exact curves when the relative variances (- - -) become too large. Nevertheless, they describe the average long-term behavior properly.

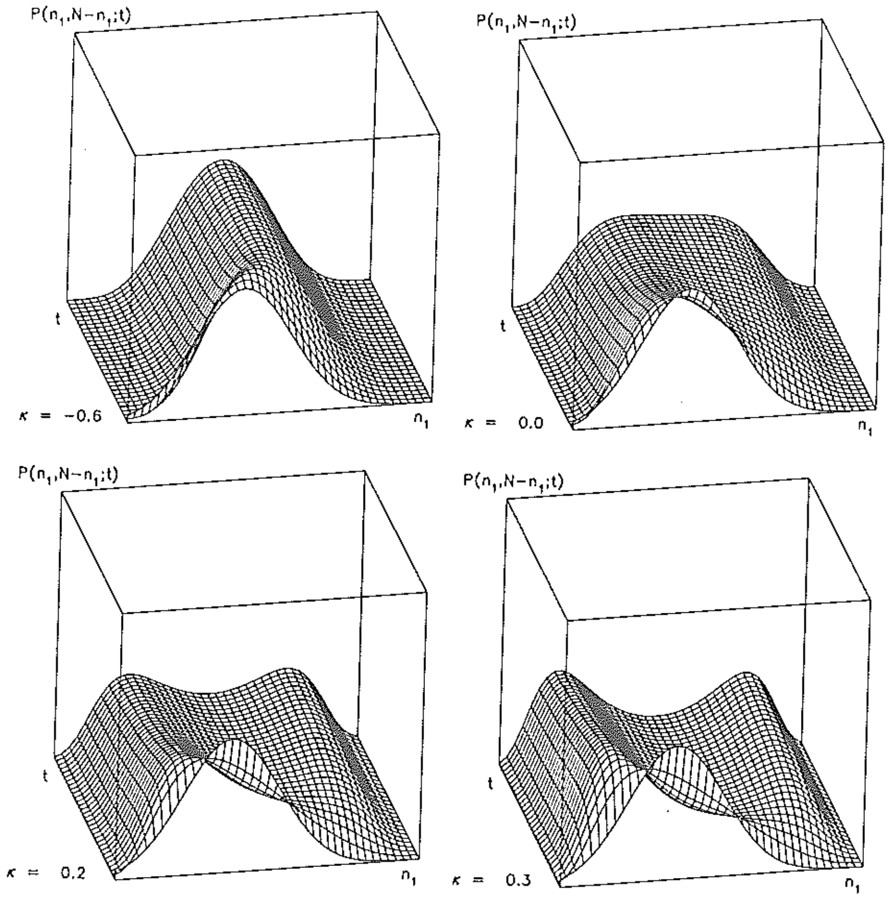

Figure 3: Probability distribution $P(\vec{n},t) = P(n_1, N - n_1; t)$ of the socioconfiguration $\vec{n}$ according to the modified stochastic game dynamical equations.

Dynamics with expectations: The decisions of individuals are often influenced by their *expectations* $\langle E_x \rangle^*_{t'}$ about the success of a strategy x at future times $t' > t$. These will base on some kind of extrapolation of past experiences with the success of x. If expected payoffs at future times t' were to be weighted exponentially by their distance $(t' - t)$ from the present time t, one would set[10]

$$\langle E_x \rangle_t = \frac{1}{T} \int_t^\infty dt' \, \langle E_x \rangle^*_{t'} \, \exp\left(\frac{t' - t}{T} \right) . \tag{35}$$

Other kinds of pair interactions: Apart from imitative behavior, individuals also sometimes show an *avoidance behavior*

$$x + x \to y + x , \tag{36}$$

especially if they dislike their interaction partner (so-called 'snob effect'). This can be taken into account by an additonal contribution to the individual interaction rates:

$$w(y \to x; \vec{n}) = w_1(y \to x) + w_2(y \to x) \, n_x + w_3(y \to x) \, n_y , \tag{37}$$

where w_3 denotes the *avoidance rate*.

Several subpopulations: Sometimes one has to distinguish different *subpopulations* a, i.e. different kinds of individuals. This is necessary if not all individuals have the same set S of strategies.[11] A similar thing holds if the social system considered consists of competing groups, where only individuals of the same group behave cooperatively. The generalized behavioral equations are[12]

$$\frac{dp^a_x(t)}{dt} = \sum_y \left[p^a_y(t) w^a(y \to x; \langle \vec{n} \rangle_t) - p^a_x(t) w^a(x \to y; \langle \vec{n} \rangle_t) \right] \tag{38}$$

with individual interaction rates of the form

$$w^a(y \to x; \vec{n}) = w^a_1(y \to x) + \sum_b \left[w^{ab}_2(y \to x) \, n^b_x + w^{ab}_3(y \to x) \, n^b_y \right] . \tag{39}$$

Inclusion of memory effects: If the strategy distribution at past times $t' < t$ influences present decisions in a non-Markovian way, the approximate mean value equations have the form

$$\frac{dp^a_x(t)}{dt} = \sum_y \int_{-\infty}^t dt' \, \left[p^a_y(t') w^a_{t-t'}(y \to x; \langle \vec{n} \rangle_{t'}) - p^a_x(t') w^a_{t-t'}(x \to y; \langle \vec{n} \rangle_{t'}) \right] . \tag{40}$$

For example, in cases of an exponentially decaying memory, one would have

$$w^a_{t-t'}(x \to y; \langle \vec{n} \rangle_{t'}) = w^a(x \to y; \langle \vec{n} \rangle_{t'}) \, \frac{1}{\tau} \, \exp\left(\frac{t - t'}{\tau} \right) . \tag{41}$$

VIII. SUMMARY AND CONCLUSIONS

We have found a microscopic foundation for the game dynamical equations which are based on a certain kind of imitative behavior. Moreover, a stochastic version of evolutionary game theory has been formulated. It allowed us to understand the self-organization of social conventions as a phase transition which is related with symmetry breaking. Moreover, we have seen that the game dynamical equations correspond to approximate mean value equations. Normally, they agree with the mean-value equations of stochastic game theory for a certain time period only, which can be determined by calculating the relative variances. For an improved description of the average system behavior we have derived corrected mean value equations which require the solution of additional covariance equations.

The interpretation of the game dynamical equations proceeds by reformulating these in terms of a *social force model,*[13] assuming a continuous strategy set:

$$\frac{dx_\alpha(t)}{dt} = f_1(x_\alpha) + \sum_{\beta(\neq\alpha)} f_2(x_\alpha, x_\beta) + \text{fluctuations}\,. \qquad (42)$$

The force term

$$f_1(x_\alpha) = \int dx\,(x - x_\alpha)\,w_1(x_\alpha \to x) \qquad (43)$$

delineates spontaneous strategy changes by individual α, whereas

$$\begin{aligned} f_2(x_\alpha, x_\beta) &= (x_\beta - x_\alpha)\,w_2(x_\alpha \to x_\beta) \\ &+ \int dx\,(x - x_\alpha)\,w_3(x_\alpha \to x)\,\delta(x_\alpha - x_\beta) \end{aligned} \qquad (44)$$

is the *interaction force* which originates from individual β and influences individual α. Here, $\delta(x - y)$ denotes *Dirac's delta function* (which yields a contribution for $x = y$ only). According to (42), the game dynamical equations describe the *most probable strategy changes* rather than the average (representative) evolution of a social system. Therefore, they neglect the effects of fluctuations on the system behavior.

A more detailled discussion of the results presented in this paper is available elsewhere[12,13].

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8. For illustrative reasons, a small number of individuals ($N = 40$) and a broad initial probability distribution have been chosen. In each picture, the box is twice as high as the maximal occuring value of the probability.
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