5 A MATHEMATICAL MODEL OF GROUP DYNAMICS INCLUDING THE EFFECTS OF SOLIDARITY

WOLFGANG WEIDLICH and DIRK HELBING

Institut für Theoretische Physik, Universität Stuttgart

1. INTRODUCTION

The dynamics of groups under inclusion of the aspect of solidarity can be considered to be one of the most general and fundamental problems of sociology. Society, being one of the main subjects of sociology, is constituted by interacting human individuals. However, they are not an amorphous crowd. The interaction "forces" between the individuals generated by material, emotional, and mental needs and desires with respect to different aspects of life lead to a *selforganization of structures* within society. These structures take the form of *groups* (i.e. ensembles of individuals with certain common objectives). Each individual will in general *simultaneously* belong to *several groups* taking the role of a nominal or leading member in each of them.

It turns out that an indispensable element of the formation of groups is the *emerging solidarity* which provides a kind of glue between their members. The *theory of group dynamics*, in which the element of *solidarity* therefore should play the role of a *central concept*, intends to understand and to formally describe not only the *stationary structure*, but also the *rise*, *evolution*, and *decay* of social groups.

The authors Fararo and Doreian (1995) of the introductory chapter to this book have given a comprehensive overview of the extensive literature already devoted to this problem.

Three main, of course intertwined, kinds of problems can be distinguished in this chapter:

1. the *problems on the micro-level concerning individuals*, including their motivations, emotions, beliefs, activities, and forces,

2. the *problems on the macro-level concerning structures,* i.e. which part-whole relations, common interests and objectives, sanction mechanisms effecting compliance with obligations, feelings of loyalty, and identifications with a collectivity, etc. finally produce the *tie of solidarity* leading to the formation of groups, and
3. *problems referring to the operationalization* of these *qualitatively* formulated concepts in a *quantitative* framework capable of capturing structural as well as dynamic aspects in a *mathematical form.*

Our contribution focusses on the third problem, i.e. how to set up a sufficiently general mathematical model of group formation and evolution for a system of interacting and competing groups under inclusion of a quantitative concept of *solidarity*. It was our intention to explicitly incorporate as many ideas about group formation as possible from the literature analyzed in the above-mentioned chapter. This was much simplified by the fact that these ideas seem not to have a mutually exclusive but rather a complementary character.

Before proceeding to the details of our model design we shall now discuss some general problems of model construction implied in the present context.

(a) *The wholeness of model construction.* Because of the strong interrelatedness of all concepts in the context of group formation it seems impossible to construct a model part by part, in a piecemeal way. Instead, theoretical coherence corresponding to the correlation of all factors involved can only be reached by constructing the whole mathematical framework *at once*. Therefore also the *empirical validation* cannot take place piece by piece but by verifying the outcomes of the whole model in as many independent cases as possible where empirical measurements are feasible.

(b) *Operational and hidden variables.* Any mathematical model must obviously contain a set of "*operational variables*" which by definition are *directly measurable,* i.e. accessible for empirical validation. But *not all* relevant variables *need* to be operational. In order to obtain a coherent theoretical and interpretational conceptualization it may turn out to be necessary to introduce "*hidden variables*" whose values cannot be measured directly but which are *indirectly* measurable by observing their effects on the operational variables in the dynamic process.

(c) *The problem of interpretational ambiguity.* The mathematical model for a self-contained set of (operational and/or hidden) key-variables may have two or more possible *qualitative* interpretations. That means, more than one behavioral motivation may lead to the same group dynamics. Therefore the full wealth of qualitatively different descriptions of attitudes can in general *not* be depicted *one to one* in the mathematical framework. Instead, the mathematical model provides a *projection* (in the sense of a *homomorphism* rather than an *isomorphism*) of the full reality into a restricted representation by few key-variables (see Helbing, 1995, p. 14, 127f).

(d) *Fictitious and real scenarios and their relation to the verification process.* Every sufficiently comprehensive mathematical model contains a set of "*trend parameters*" which denote the parameters in the evolution equations of the relevant variables. Even if the *meaning* of the trend parameters is clear, their numerical *value* remains *open* for the first. Each set of trend parameters within the admissible domain and the corresponding simulation of the dynamics leads to a scenario which is *theoretically possible* within the frame of the model. However, most of these scenarios are *fictitious*, whereas the few *actually observable scenarios* should be seen as embedded into the vast manifold of theoretically possible but *fictitious scenarios*. The selection of *realistic* scenarios from all possible scenarios simultaneously leads to a *validation* of the model (if realistic scenarios are contained in the manifold of the theoretically possible ones) and to an appropriate *calibration* of the trend parameters. But also fictitious scenarios belonging to extreme choices of trend parameters and correspondingly biased motivation structures can be of high importance: They can serve for a study of the effect of artificially exaggerated motivations on the dynamics of the system.

The sections of this chapter are organized as follows: In Section 2 the design principles of the model will be explained in *qualitative terms*. Section 3 will then contain the explicit mathematical formulation of the model, beginning with the definition, explanation, and interpretation of the relevant key-variables, trend-functions, and trend-parameters. Further we will set up coupled differential equations for the evolution of these variables. In Section 4 selected scenarios will be simulated as special solutions of the model equations. Each scenario will be accompanied by a sociological interpretation. The simulations can of course not exhaust the content of the model and should be considered as a first step towards its numerical investigation. Finally, Section 5 is an attempt to establish a relation between our model and selected literature in this field.

2. THE DESIGN PRINCIPLES OF THE MODEL

In this section we shall exhibit the problems and develop the design principles of our model in a verbal non-mathematical form but with the intention to prepare its mathematical formulation in the next section.

The purpose of the model is to give an insight into the dynamics of the evolution of groups and their competitive interaction on the macroscopic level. The dynamics will imply stages of rise, evolution, stationarity, and decay.

Studying the literature (see also the discussion in Section 5) it becomes apparent that group structure and dynamics is considered on different scales and from different perspectives.

Some approaches focus on the more microscopic level of individual activities and interactions which represent the daily routine of group life and reproduction. We denote the fast and fluctuating variables belonging to this scale as *micro-variables*. These include the *network-variables* which describe the activities within the social network (*network activities*).

Other approaches focus on the slowly varying collective variables on the macro-level which characterize and dominate the global structure and evolution of a system of interacting groups. We denote these slow macro-variables as *key-variables* or *order-parameters*.

Evidently all variables are interconnected. In particular, the key-variables are somehow composed of the network-variables. Therefore a full theoretical description of group dynamics should comprise both the microscopic network-variables and the key-variables, however, at the cost of immense complexity and tangledness.

Instead we shall restrict ourselves to a formal description of the *macro-level* taking into account a set of relatively few *key-variables* only.

This procedure must be *justified* because it implicitly assumes that the key-variables exhibit a *quasi-autonomous self-contained sub-dynamics of their own*, without explicitly taking into account their connection with the microscopic network-variables.

In cases where a complete set of equations for both micro- and macro-variables is available, as it happens for many physico-chemical systems, such a justification can be given. It follows from the so-called "*slaving principle*" discovered by Haken (1977). Its qualitative meaning can be easily understood: The fast-moving micro-variables nearly instantaneously adapt to their momentary equilibrium values, which gradually vary with the slow temporal changes of the macroscopic order-parameters. By this, the dynamics of the micro-variables is *already determined* by the slowly changing order-parameters. Consequently, the micro-variables can be mathematically *eliminated* by expressing them in dependence on the macro-variables. There remains a reduced *set of autonomous dynamic evolution equations for the order-parameters only* into which the micro-variables enter merely implicitly via the elimination procedure.

Can this procedure be transferred to sociodynamics, in particular to our present problem?

On the one hand, it can be well substantiated that the relation between the fast network-variables of daily routine and the slow key-variables of global group structure is analogous to the one between the fast micro-variables and the slow macro-variables in physico-chemical systems. Indeed, the dynamics of the fast network-variables is constrained, guided, and dominated by the (slowly evolving) structure of the group. Presuming that the momentary global structure of the group is a reflection of the daily network activities, which, however, do not appear explicitly, the main remaining problem is the dynamics of the slowly evolving key-variables characterizing the macro-structure of the group.

On the other hand, there is no system of equations available at all for which an elimination procedure via the "slaving principle" could be applied. Therefore the search for a sufficiently comprehensive and self-contained set of key-variables and the introduction of their dynamics will base on a combination of observation and intuition.

In our modelling procedure we restrict ourselves to a deterministic (not stochastic) descriptive level, which is equivalent to a consideration of the mean-values of the key-variables neglecting their probabilistic fluctuations. This is done in order to avoid a too high complexity which could not be exhausted anyway. Our approximation can be justified if the probabilistic fluctuations of the considered variables are small. This holds if the considered groups are large enough so that the deviations from the mean-values are small. The approximation may however become problematic for groups in which the number of members is small.

An extension to stochastic equations which take into account fluctuations is possible. The equations for the key-variables then appear as the approximate mean-value equations of the corresponding stochastic equations; Weidlich and Haag (1983), Weidlich (1991, 1994) and Helbing (1995).

Let us now go over to the details of model design.

The Key-Variables of the Model

We have now to search for a set of adequate key-variables dominating the structure and evolution of groups on the macro-level. Our conclusion is that two kinds of variables are indispensable. The first kind are the.

Collective Personal Variables

The members of a group do not form a homogeneous crowd. A group rather develops a *hierarchy of subgroups* the members of which possess *different degrees of influence, responsibility, and obligation*. The hierarchy may comprise few or many organizational levels reaching from nominal membership to leadership. In any case, an essential structural feature of a group is the number of levels of influence and obligation and the distribution of its members over these hierarchical subgroups. The numbers of group members on the different hierarchical levels can be combined in a vector that is denoted as *group configuration*. Here, the group configuration summarizes a set of *slow time-dependent collective personal variables* that partially characterize the group structure and its dynamics at the macro-level.

If the activities, responsibilities and obligations as well as the attitudes and the mentality belonging to each organizational level were described in detail, we would have a description of the *micro-level of network-activities* of the group down to the behavior of a single individual member. These micro-activities however *do not explicitly appear* in the macro-variables of the group configuration which are nevertheless a *resultant* of the daily network activities.

Transpersonal Variables

The "*groupness of a group*" is only partially captured by variables like those comprised by the group configuration. A group is led by ideas and visions which sometimes even form a *coherent ideology* with respect to the common purposes and objectives of the group. These ideas and visions provide a kind of *glue* keeping the group together and establishing a feeling of *identification* with the group among its members. The *intensity* of this sentiment may reach from "feeling at home in the group" up to enthusiasm and fanaticism.

The togetherness and common bond of group members coming about in this way creates an *indirect* relation between them because each member is indirectly tied to the others by adhering to the same group ideology. This relation *differs* from the one coming about by *direct* interaction (e.g. by imitation, persuasion, urging to activities) between each pair of members of the group.

In general, *direct* as well as *indirect* relations will contribute to the formation of the groupness of a group but the relative weight of both may vary considerably. In groups of political extremists or religious sects the fanatic belief in the group ideology will prevail, whereas in clubs or associations devoted to sports or arts the interindividual partnership will prevail.

The aspects of group ideology are, on the one hand, established and sustained by all individual members. On the other hand, these ideas act back on the members by inspiring them and urging them to become active supporters of the group. But just by being supported by *all* and, vice versa, by inspiring *all* members of the group, the aspects of group ideology take on a *transpersonal quality* since they *cannot* be attributed and allocated to any particular person. Therefore it is adequate to treat them as *entities on their own*, i.e. as conceptual realities of separate and distinct existence, and to endow them with their *own dynamics*.

Our conclusion is that consequently one should attribute to each group a *set of transpersonal variables* satisfying their own dynamic equations (which are coupled to the variables of the group configuration) and describing substantial aspects of the group ideology and its interaction with the members. In a *minimal model* one should focus on one transpersonal variable which *abstracts* from the contents and colors of a specific ideology and simply provides a measure for the *intensity* of the *emotional affection and adherence* produced by the group ideology among the members of the group. *This central transpersonal variable is denoted as the group solidarity.*

At this stage we will postpone the question whether the group solidarity should be considered as an *operational* or a *hidden* variable (see Section 1), i.e. whether it can be measured directly or only indirectly.

The *full set* of key-variables of a minimal model of interacting groups thus will consist of the *group configuration* and the *group solidarity* for each group.

The Elementary "Driving Forces" of the Dynamics

The main problem now consists in finding the elementary driving forces responsible for changes of the key-variables in order to insert these in a consistent, self-contained system of evolution equations.

First we note that a stationary state of a system of groups will consist of the stationary numbers of non-members and members of all groups on the different levels of hierarchy and of the stationary values of the variables of group solidarity. However, the knowledge of the stationary state itself does not contain much information about the causes of its generation; in other words, there lacks a process conceptualization. In order to learn more about these causes we must try to understand how the *changes of the key-variables*, i.e. the changes of the numbers of the group configuration and the increases or decreases of the solidarity variables come about.

The change of the group configuration, i.e. the change of the numbers of members of a group occupying certain roles and levels in the hierarchy is evidently due to the *transition of individuals* from one role to another (for instance the transition from non-membership to the status of a nominal member of a group, or the transition from the status of a nominal member to that of a leading member).

At this point the dynamics of those key-variables which describe the group configuration is influenced by the decision-making of individuals who occupy a role which they want to change. The crucial "*driving force*" behind the decision to make such a transition is a *conditional motivation* to adopt – *ceteris paribus* – a certain new role in the group configuration, depending on the respective role one has occupied before.

The *mathematical formulation* of this matter, which is the subject of Section 3, will therefore consist in introducing *transition rates* (per unit of time) for individuals to move from old to new roles within the group configuration. These transition rates will depend on *conditional motivations*. And, since these motivations arise under the given situation, they will thus have to be appropriately chosen *functions* of all *key-variables* which characterize that situation. The so-defined transition-rates then immediately lead to the formulation of the *equations of motion for the collective personal variables of the group configuration*. They have the form of *generalized migratory equations*, Weidlich and Haag (1988) and Weidlich (1991), where of course the migration does not take place between locations as in conventional population dynamics but between status roles within the group configuration.

Finally we have to consider the *dynamics of the solidarity variable* of each group. It is clear that the evolution of this variable is coupled to collective activities and attitudes of all members of the own group and perhaps also of the competing ones. Since the solidarity is a transpersonal variable, only the *superposition* of individual activities and attitudes, i.e. only *collective efforts and moods* will lead to an increase or decrease of the quantified measure of solidarity.

Two main kinds of influences on the dynamics of the solidarity variable are conceivable:

Firstly, within a group constructive collective efforts of its group members towards *consolidation* of the group identity will lead to an *enhancement* of the amount of solidarity in that group whereas, on the other hand, members of other groups might make destructive efforts in order to disintegrate the identity of the competing group and thus possibly cause a diminution of solidarity. Of course, a superposition of the *constructive* and the *destructive* efforts will take place.

Secondly, there will exist a *saturation effect* prohibiting an unlimited increase of solidarity even in the absence of interfering competing groups. This saturation is due to a *limited total capacity* of every individual to engage in the pursuit of ideals such as the identification with a group. If the individual is simultaneously a member of several groups (belonging to different "dimensions of life"), he/she will have to divide his/her *receptivity* for group ideology and solidarity between these groups. The disappointment-creating factors like the free-rider phenomenon depend on the size of the group and so does the saturation level. Hence a *limit of group size* is connected with a *limited capacity for solidarity*.

Taking into account these two main influences one is led to setting up an equation of motion for the dynamics of solidarity in each group, containing a *growth term* and a *saturation term* in the form of a generalized logistic equation. However, the growth term and the saturation term will have to be appropriately chosen functions of the group configuration. Hence the dynamics of solidarity is coupled to the values of the other (collective personal) key-variables. We have seen above that the inverse is also true: The dynamics of the group configuration is coupled to the values of the solidarity variable because the conditional motivations depend on this variable, too.

In total there arises a system of *coupled autonomous differential equations of motion* for the components of the group configuration describing the evolution of the *size* and *composition* of a group, together with equations of motion for the evolution of *solidarity* within each group.

Concluding this section we stress once more that the quantitative model now to be set up in mathematical detail in the following section refers to macro-variables and not to micro-variables. The detailed behavior of the latter is therefore not explicitly reflected in this model. It is, however, presumed implicitly (and could be substantiated by a micro-theoretical approach) that the basic dynamics of the microscopic variables carries, supports, and sustains the macro-structure and macro-dynamics captured in this model.

3. THE MATHEMATICAL FORM OF THE MODEL

The design principles of Section 2 have now to be cast in concrete mathematical form.

In Section 3.1 we begin with some preliminary remarks which will prove useful in the course of the construction. Thereupon we follow the procedure indicated in Section 2.

The key-variables as such and some derivative variables are introduced in Section 3.2. In Section 3.3 the construction elements of the dynamics are introduced and interpreted. At the end of this chapter all key-variables, trend functions, and trend parameters together with their names, brief interpretations, and defining equations are summarized in tables. The pre-requisites built up in Sections 3.2 and 3.3 are then used in Section 3.4 to set up the system of evolution equations for the key-variables. In Section 3.5 problems connected with the solution of these equations are discussed. In particular we will see that the equations may possess one or more stationary solutions which are approached for $t \to \infty$.

3.1. Preliminary Remarks

At first we treat the question of the *range of interpretations*, which was already discussed in the introduction pointing at the possible ambiguity of interpretations of one and the same formal model. In Section 2 we have preferred a "psycho-social" vocabulary and have avoided interpretations with an economic touch. Now we shall *simultaneously* make use of both *psycho-social* and *economic* interpretations, i.e. of more value-oriented idealistic and more cost-oriented materialistic argumentations. For instance, in speaking about the positive and negative aspects of a certain status level we shall use *psycho-social terms* like advantage, gratification, satisfaction and frustration, burden of obligations, disappointment, etc. as well as *economic terms* like utility, benefit, payoff and costs, charges, fees, etc. Although some sociologists may refuse to compare economic matters of "filthy lucre" with psycho-social matters like values and idealistic commitment we think that it is not only *possible* but even *inevitable* in any quantitative sociodynamic theory to compare and to relate both interpretational aspects. If, for instance, in a decision process to change from one status level to another the immaterial and the material aspects play a simultaneous role, in any quantitative theory suppositions must be made about the quantitative amounts and the ratio of intensities of both motivations.

Secondly, we will now make a simplification by *restricting our group dynamic formulation to one "dimension of life"*. By "dimension" we mean one sector of social life belonging either to the religious, cultural, political, economic, or leisure sphere. Each individual may simultaneously be a member of a group in each of these sectors. But in *one* sector it can be assumed that the groups are *non-overlapping* (see Figure 5.1) because individuals will normally be members of at most *one group* in each sector. For instance, memberships in political parties are mutually exclusive. Neglecting interactions between the different sectors (and of memberships in groups belonging to different sectors) we consider one arbitrary sector only and assume that only non-overlapping groups exist in that sector.

Thirdly, we will make use of the following *notation of variables*, which is also summarized at the end of Section 3.3: *Key-variables* are denoted by *capitals; trend-functions* (depending on key-variables) and *trend-parameters* (constant coefficients) are denoted by *small italics*. All these quantities are dimensionless plain numbers. *Rates*, however, which have the dimension 1/(unit of time) are denoted by *greek letters*: their value determines the speed, i.e. the time scale on which the evolution takes place. This time scale will range from months to years or even decades since we consider the evolution of slowly varying macroscopic key-variables.

3.2. The Key-Variables

Let us consider G non-overlapping groups $G_i, G_j, \ldots, i, j=1,2,\ldots,G$ belonging to one sector of social life. As already indicated in Section 2, there exists a hierarchy of status levels h, $k=0,1,2,\ldots,H_i$ in each group G_i from nominal membership $h=0$ over staff levels up to the leading level $h=H_i$, where each level has benefits and advantages but also charges and obligations of its own. In each group G_i the *total*

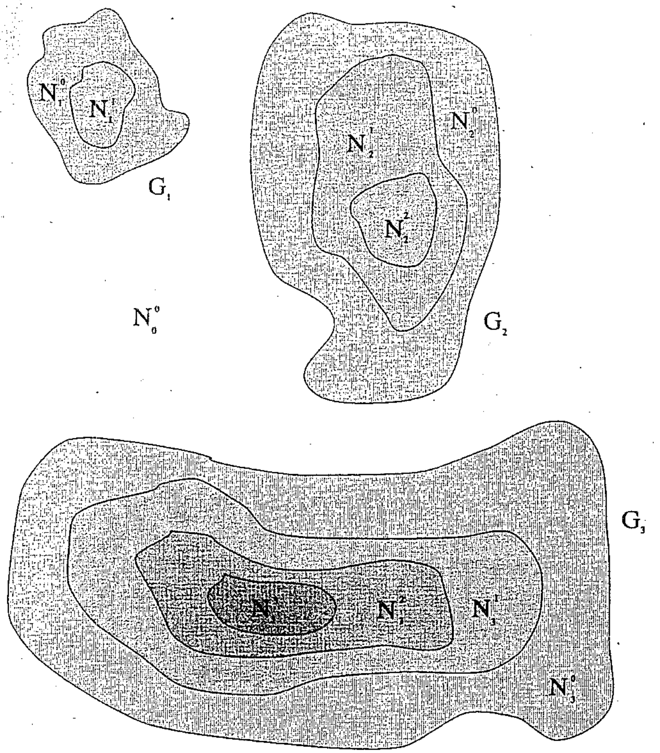

FIGURE 5.1. The picture illustrates one social sector (dimension) with three nonoverlapping groups G_1,G_2,G_3. The points of the plane represent the individuals of the total population. Their position in this abstract space has nothing to do with their location in real space. The space between the groups represents the crowd of N_0^0 non-members. G_1 has two, G_2 three, and G_3 four hierarchical subgroups corresponding to nominal up to leading members of the groups. The components $\{N_1^0,N_1^1;N_2^0,N_2^1,N_2^2;N_3^0,N_3^1,N_3^2,N_3^3\}$ of the group configuration are assigned to these subgroups. The grade of obligations and influence of the subgroups are indicated by hatchings of different densities.

number of its members N_i is distributed over the status levels h in a characteristic way. If N_i^h denotes the *number of members of* G_i *occupying the status level* h, one obtains the obvious relation

$$N_i=\sum_{h=0}^{H_i} N_i^h. \tag{1}$$

A member of group G_i with status h will be said to be in state (ih). Furthermore we introduce the number N_0^0 of individuals being

involved in none of the groups and refer to these as to the crowd of individuals. For these individuals we do not distinguish any status levels, i.e. $N_0^0=N_0$.

Since double memberships in one and the same social sector are excluded by assumption, we have the relation

$$N_0^0+\sum_{i=1}^{G} N_i=N_0^0+\sum_{i=1}^{G}\sum_{h=0}^{H_i} N_i^h=N, \tag{2}$$

where N is the total number of individuals in the social system considered. The set of (time-dependent) numbers

$$\mathbf{N}=\{N_0^0;N_1^0\dots N_1^{H_1};\dots;N_i^0\dots N_i^h\dots N_i^{H_i};\dots;N_G^0\dots N_G^{H_G}\}$$
$$=\{N_0^0;\mathbf{N}_1;\dots;\mathbf{N}_i;\dots;\mathbf{N}_G\} \tag{3}$$

is denoted as (total) *group configuration*, where $\mathbf{N}_i$ is the *partial group configuration of* G_i.

Furthermore we introduce the *shifted group configuration*

$$\mathbf{N}_{ji}^{kh}=\{N_0^0;N_1^0\dots N_1^{H_1};\dots;N_j^0\dots(N_j^k+1)\dots N_j^{H_j};$$
$$\dots;N_i^0\dots(N_i^h-1)\dots N_i^{H_i};\dots;N_G^0\dots N_G^{H_G}\} \tag{4}$$

arising from N after the transition of one individual from state (ih) into state (jk).

From the group configuration further variables can be directly derived. Evidently,

$$\sum_{k=0}^{h-1} N_i^k=\text{personnel below status } h \tag{5}$$

is the number of all subordinates of the status level h, and the *average number of subordinates of* one *member with status* h is given by

$$n_i^h=\sum_{k=0}^{h-1}\frac{N_i^k}{N_i^h}. \tag{6}$$

It is plausible to assume that the potential *influence and power* of a staff member of G_i with status h is more or less proportional to the number of members subordinate to him plus himself. Therefore we introduce the *potential influence in state* (i, h) as

$$i_i^h(\mathbf{N}_i) = n_i^h + 1 = \sum_{k=0}^{h} \frac{N_i^k}{N_i^h}. \tag{7}$$

An indicator of the *total* (internal as well as external) *potential influence* and power of group G_i can be defined as the sum of the influences i_i^h of all its members:

$$i_i(\mathbf{N}_i) = \sum_{h=0}^{H_i} N_i^h i_i^h = \sum_{h=0}^{H_i} \sum_{k=0}^{h} N_i^k. \tag{8}$$

However, a more detailed consideration leads to taking into account *saturation effects*: For instance, in large groups the potential influence i_i^h will not really grow proportionally to the number of subordinates because the intensity of direct personal interactions with these subordinates cannot be maintained if their number increases. Therefore, instead of N_i^h, we introduce saturated numbers $\tilde{N}_i^h$.

$$N_i^h \to \tilde{N}_i^h = N_{\max} \frac{N_i^h}{(N_{\max}-1)+N_i^h} = N_{\max} y_i^h. \tag{9}$$

with

$$N_{\max} > 1 \quad \text{and} \quad 0 \leqslant y_i^h = \frac{x_i^h}{1+x_i^h} \leqslant 1, \ \text{for } x_i = \frac{N_i^h}{(N_{\max}-1)} = \frac{y_i^h}{1-y_i^h}. \tag{10}$$

Evidently, the *saturated numbers* $\tilde{N}_i^h$ have the following properties:

$$\tilde{N}_i^h = N_{\max} y_i^h = \begin{cases} 0 & \text{for } N_i^h = 0, \\ 1 & \text{for } N_i^h = 1, \\ \tilde{N}_i^h \lesssim N_i^h & \text{for small } N_i^h, \\ \tilde{N}_i^h \Rightarrow N_{\max} & \text{for } N_i^h \Rightarrow \infty, \\ \tilde{N}_i^h \Rightarrow N_i^h & \text{for } N_{\max} \Rightarrow \infty. \end{cases} \tag{11}$$

It is now easy to take into account the saturation effects in (6)–(8) by going over to the *saturated potential influence in state* (ih):

$$\tilde{i}_i^h(\mathbf{N}_i) = \sum_{k=0}^{h} \frac{\tilde{N}_i^k}{\tilde{N}_i^h} \leqslant (h+1) \frac{N_{\max}}{\tilde{N}_i^h} \tag{12}$$

and the *total* (*saturated*) *potential influence*:

$$\tilde{i}_i(\mathbf{N}_i) = \sum_{h=0}^{H_i} N_i^h \tilde{i}_i^h = \sum_{h=0}^{H_i} \sum_{k=0}^{h} \tilde{N}_i^k \leqslant \frac{(H_i+1)(H_i+2)}{2} N_{\max}. \tag{13}$$

The number $N_{\max}$ appearing in $\tilde{N}_i^h$ has to be calibrated carefully because it limits the range of influence of higher status levels. It may be different in different social sectors or contexts, e.g. different for political parties and for sport clubs.

Finally, we assign a *solidarity variable* S_i to each group G_i. This variable is (see Section 2) to be interpreted as a transpersonal measure for the intensity of the emotional affection and adherence evoked in all members of the group by the common group ideology and by the group objectives.

Direct measurements of S_i are certainly difficult but not impossible. They could consist in measuring a composite indicator of reactions of members of the group (taking into account, e.g. the frequency of voluntary compliance with obligations). But independently of the question whether S_i is a directly measurable operational or a hidden variable it will play an explicit and definite role in the evolution equations as we will see in Section 3.4.

Anyway we are free to choose the domain of variation of S_i by an appropriate scaling and by confining $S_i(t)$ to a domain which it must not leave in the course of time. We choose

$$0 \leqslant S_i(t) \leqslant 1, \quad i = 1, 2, \ldots, G. \tag{14}$$

All key-variables can now be comprised in the *total key-variable configuration*

$$\begin{aligned} \mathbf{C} &= \{N_0^0; S_1, \mathbf{N}_1; \ldots; S_i, \mathbf{N}_i; \ldots; S_G; \mathbf{N}_G\} = \{\mathbf{S}, \mathbf{N}\} \\ &= \{N_0^0; \mathbf{C}_1; \ldots; \mathbf{C}_i; \ldots; \mathbf{G}_G\}, \end{aligned} \tag{15}$$

where $\mathbf{C}_i = \{S_i, \mathbf{N}_i\}$ is the *key-variable configuration of* G_i and $\mathbf{S} = \{\mathbf{S}_1, \ldots \mathbf{S}_i \ldots \mathbf{S}_G\}$. Corresponding to (4) we also introduce the *shifted key-variable configuration* $\mathbf{C}_{ji}^{kh} = \{\mathbf{S}, \mathbf{N}_{ji}^{kh}\}$.

We conclude this section with Figure 5.1 illustrating a sytem of non-overlapping groups belonging to one social dimension and possessing internal hierarchical structures. All social dimensions together could be visualized by superposing figures of all dimensions of life. Then the groups of different figures would in general overlap because individuals can simultaneously be members of different groups belonging to different social dimensions.

3.3. Construction Elements of the Dynamics

On the way towards the construction of dynamic equations for the key-variables **C** we need an *intermediate* but very important step: the introduction of certain mathematical concepts providing the *elementary components of the dynamic process*. These elementary *dynamic components* must finally be connected with the *motivations* for those decisions of individuals which lead to changes of the key-variables.

With respect to the collective personal variables **N** we proceed in three steps. First we introduce *transition rates* which describe the elementary changes of the group configuration. Second we introduce *conditional motivation potentials* which are measures of the readiness of the individuals to perform transitions. Third we establish the relation between the transition rates and the motivation potentials.

(a) Transition Rates between States of Individuals

The elementary step leading to a change of the group configuration is the transition of a member of group G_i with status h, i.e. of an individual in state (ih), into state (jk) by becoming a member of group G_j with status k. Such elementary steps take place stochastically, i.e. the sequence of such steps is a random process. However, going over to *mean-values*, i.e. considering an ensemble of individuals doing such steps in the same environmental situation characterized by the key-variables **C**, one can introduce the well-defined quantity

$$\nu(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C}) = \text{total transition rate}, \qquad (16)$$

which is by definition, the fraction of individuals originally in state (ih) who per unit of time go over into the new state (jk).

The total transition rate is not only a function of the initial state (ih) and final state (jk) but also of the whole situation (expressed by the key-configuration **C**) *before* the transition and the whole situation (expressed by the shifted key configuration $\mathbf{C}_{ji}^{kh}$) *after* the transition.

We now anticipate that two different processes contribute to the total transition rate: On the one hand, there exist transitions from (ih) to (jk) which are *indirectly* motivated by the general mood of the individuals in state (ih) originating from the global situation. On the other hand, there exist transitions from (ih) to (jk) which are *directly* induced by persuasive activities of the members of group G_j.

Hence it is plain to decompose the total transition rate into contributions arising from indirect and direct interactions:

$$\nu(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C}) = \nu_I(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C}) + \nu_D(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C}). \qquad (17)$$

(b) Conditional Motivation Potentials

Up to now the transition rate (16) is a *purely mathematical concept* and must in the following be specified in a sociological way. This leads us to the question how the *motivations* of the transition-making individuals can be cast into a mathematical form and how this form depends on the key-variables characterizing the situation within the system of competing groups.

We tackle this problem by introducing a quantitative measure for the readiness of the individual to change into state (jk) given that he/she is in state (ih) before the transition. This measure shall be called the *conditional motivation potential*:

$$m(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C}) = \text{conditional motivation potential}. \qquad (18)$$

For the time being we *postulate* only some general properties for the conditional motivation potential: It is a real function of the initial and final state and of the key-variables characterizing the situation. It can vary in the range from $-\infty$ to $+\infty$:

$$-\infty < m(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C}) < +\infty. \qquad (19)$$

If the final state (jk) is more attractive than the initial state (ih), $m(\ldots|\ldots)$ will be positive. In the opposite case $m(\ldots|\ldots)$ will be negative. Furthermore, $m(\ldots|\ldots)$ will increase monotonously with the attractiveness of the final state (jk) and decrease monotonously with the attractiveness of the initial state (ih). So far we have neglected transaction costs.

For a given attractiveness of the initial and final state $m(\ldots|\ldots)$ should decrease with growing (economic) *transaction costs* and with a growing (psychological) *resistance* against a transition from state (ih) to state (jk).

These requirements for the conditional motivation potential suggest the following functional form:

$$m(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C}) = u_j^k(\mathbf{C}_{ji}^{kh}) - u_i^h(\mathbf{C}) - t_{ji}^{kh}, \qquad (20)$$

where $u_j^k(\mathbf{C}_{ji}^{kh})$ and $u_i^h(\mathbf{C})$ are the *net utilities* of the state (jk) after the transition and of the state (ih) before it, respectively, and t_{ji}^{kh} is a term comprising the *transaction costs* (economic costs as well as the effect of psychological resistance) related to the transition from (ih) to (jk).

The net utility of a state (ih) – and analogously of any other state (jk) – measures the attractiveness of this state (ih) and varies within the same range as the conditional motivation potential, namely from $-\infty$ to $+\infty$. It describes the relative weight of *benefits* b_i^h (satisfactory, gratifying aspects) and *costs* c_i^h (frustrating, disappointing aspects) of state (ih). Therefore the following form of u_i^h (and analogously of any other utility u_j^k) is proposed:

$$u_i^h = b_i^h - c_i^h. \tag{21}$$

Benefits and costs can again be decomposed into sub-terms of different motivational origin:

$$b_i^h = p_i^h + f_i, \tag{22}$$

$$c_i^h = o_i^h + q_i. \tag{23}$$

For the subterms we choose the following terminology:

p_i^h = payoff, f_i = faith confirmation,

o_i^h = obligations, q_i = contributions.

The meaning of these four terms will now be shortly explained.

We begin with the *cost terms*: The *obligations* o_i^h comprise the burdens of duties and responsibilities as well as the work related to members of group G_i with status h. The obligations have a predominantly immaterial character. Compliance with these obligations can be partially enforced (by sanctions) and partially voluntary altruistic.

The *contributions* q_i have a material as well as immaterial character and consist of regular payments (e.g. membership fees and grants providing the financial support of the group) but also of personal initiatives.

Turning now to the *benefit terms*, the *payoff* term p_i^h consists of partially material, partially immaterial contributions. It comprises material rewards (status-dependent remunerations) as well as status-, influence- and solidarity-dependent advantages, honors, and satisfactions.

The *faith confirmation* term f_i describes the immaterial satisfaction by receiving a reconfirmation of the own ideas and beliefs due to being a member of group G_i. However, faith confirmation may turn into *opinion pressure*, and voluntary consent into an enforced one if the entrainment forces into the group ideology become too strong and penetrant. (In political parties this would mean the mutation from a liberal to a totalitarian structure (see also Weidlich and Haag, 1983; Weidlich, 1994)).

Finally, the *transaction costs* can also be decomposed in a plausible manner:

$$t_{ji}^{kh} = a_j^k + l_i^h, \tag{24}$$

where l_i^h are the "*leaving costs*" and a_j^k the "*admission costs*".

The *leaving costs* are often primarily immaterial losses and consist of losing and breaking off old contacts when withdrawing from a previously joined group G_i.

The *admission costs* consist of an entrance fee and of the efforts necessary to be accepted in a new group G_j. Both terms together diminish the motivation to change from the previous state (ih) to a new state (jk) even if the utility of (jk) is higher than the utility of (ih).

Inserting Eq. (21) together with (22)–(24) into (20) one obtains a *decomposed form of the conditional motivation potential* in which each term has a definite meaning and interpretation.

The next task is to calibrate the trend-parameters l_i^h, a_j^k and, even more important, to propose plausible forms for $o_i^h(\mathbf{C})$, $q_i(\mathbf{C})$, $p_i^h(\mathbf{C})$ and $f_i(\mathbf{C})$ as *functions of the key-variables.* We begin with the contributions q_i which are assumed to be independent of status h. They could also be considered as function of $\mathbf{C}_i$, e.g. as function of the group size $(\sim N_i)$ and of the solidarity S_i. But for the first we assume that this dependence is relatively weak and treat q_i as a *trend parameter* which only has to be appropriately calibrated.

Continuing with the *obligations* o_i^h the following form seems persuasive:

$$o_i^h = o_i h^{d_i}. \tag{25}$$

The meaning of this formula is the following: For members without any functions $(h=0)$ the costs (23) correspond to the contributions q_i only, i.e. $o_i^0 = 0$. Obligations for members with higher status $h > 0$ grow with a certain power $d_i \geqslant 0$ of the status level h, denoted as *exponent of obligations.*

The payoff p_i^h seems to be more complicated. We arrive at the following formula:

$$p_i^h(\mathbf{C}) = N_i q_i S_i (g_0 + g_1 \tilde{i}_i S_i) \frac{r_i^h}{\sum_{k=0}^{H_i} r_i^k N_i^k} \tag{26}$$

with

$$r_i^h = q_i + r_i o_i^h = q_i + r_i o_i h^{d_i}, \tag{27}$$

which has the following interpretation: $N_i q_i$ is the sum of all (material and immaterial) contributions available for group G_i. The solidarity factor $S_i \leqslant 1$ provides a measure for the reliability with which the members make their contributions. The next factor $(g_0 + g_1 \tilde{\iota}_i S_i)$ contains modifications of a pure redistribution of contributions q_i. The first term $g_0 < 1$ expresses the *diminution of the payoff by administration costs*. The second term represents an immaterial or material increase of the payoff by group activities. This term is proportional to the potential (saturated) total influence $\tilde{\iota}_i$ and to the solidarity S_i putting this influence into effect; the coefficient g_1 calibrates the *effective total influence* $\tilde{\iota}_i S_i$. The last factor, namely the fraction containing the *share coefficients* r_i^h, r_i^k is denoted as the *payoff distribution* and it determines the share received by a member of group G_i with status h. The *share coefficient* r_i^h itself consists of the status-independent term q_i and a term proportional to the obligations, $r_i o_i^h$, where r_i is a *reward coefficient.*

The payoff equation (26) has the following obvious implication:

$$p_i^h / p_i^k = r_i^h / r_i^k. \tag{28}$$

Furthermore, the total payoff for all members of G_i is

$$\sum_{k=0}^{H_i} N_i^k p_i^k = N_i q_i S_i (g_0 + g_1 \tilde{\iota}_i S_i). \tag{29}$$

Finally, the *faith confirmation* term is assumed to have the form

$$f_i(\mathbf{C}) = \sum_{j=1}^{G} w_{ij} \tilde{\iota}_j S_j, \tag{30}$$

which can be validated as follows: The *faith feedback coefficients* w_{ij} represent the strength of the effective total influence $\tilde{\iota}_j S_j$ of group G_j on the faith in the values of group G_i. The term $w_{ij} \tilde{\iota}_j S_j$ with $j = i$ and $w_{ii} > 0$ describes a positive feedback effect of the total effective (saturated) influence $\tilde{\iota}_i$ of group G_i on the firm belief into the values of the own group G_i. This reconfirmed belief is considered as a positive part of the utility u_i^h of state (ih). The influence of competing groups on the attitudes within G_i can be constructive ($w_{ij} > 0$) but also destructive ($w_{ij} < 0$) and thus diminish f_i. In this way a competing group can have a positive or a negative influence on the utilities of another group.

Inserting Eqs. (25), (26) and (30) into formula (21) and making use of Eqs. (22) and (23), one obtains the explicit expression

$$u_i^h(\mathbf{C}) = (p_i^h(\mathbf{C}) + f_i(\mathbf{C})) - (o_i^h + q_i) \tag{31}$$

for the net utility as a function of the key-variables. Therefore also the conditional motivation potential (20) has now become an explicit function of the key-variables.

(c) *Representation of Transition Rates in Terms of Motivation Potentials*

Our last step consists in establishing a connection between the conditional motivation potential and the (indirect and direct) transition rates. In this way also the transition rates become explicit functions of the key-variables.

According to their definition, the indirect and direct transition rates $\nu_I(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C})$ and $\nu_D(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C})$, respectively, are positive definite quantities describing the frequency of transitions of individuals from an initial state (ih) to a state (jk) in dependence on these states and on the global situation $\mathbf{C}$ among the competing groups.

It is natural that these transition rates are higher the more attractive is the final state (jk) compared to the initial state (ih), and that they are smaller the higher are the transaction costs or the psychological resistance for a transition from (ih) to (jk).

In the same manner, the values of the conditional motivation potential (which, however, ranges from $-\infty$ to $+\infty$) *increase* with growing difference between the attractiveness of the final and initial state, and *decrease* with growing transaction costs.

Therefore it is *clear* that there must exist a monotonous functional relation between the (indirect and direct) transition rates and the conditional motivation potential. The simplest (and, as it will turn out, the most plausible) monotonous function transforming the domain $-\infty < m(\ldots|\ldots) < +\infty$ of the conditional motivation potential into the positive domain $\nu_I, \nu_D > 0$ of the transition rates is the exponential function. Therefore we postulate the proportionality relations:

$$\begin{aligned} \nu_I(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C}) &\sim \exp[m(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C})], \\ \nu_D(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C}) &\sim \exp[m(jk; \mathbf{C}_{ji}^{kh} | ih; \mathbf{C})]. \end{aligned} \tag{32}$$

Whereas the bias in favor of either the final state (jk) or the initial state (ih) is already sufficiently taken into account by $m(jk; \mathbf{C}_{ji}^{kh} | ih, \mathbf{C})$, the special effects of persuasion activities of members of the envisaged group G_j favoring the transition to state (jk) must be described by an additional factor regarding the direct transition rate ν_D. This *persuasion activity* is assumed to have the form

$$e_j \sum_{k=0}^{H_j} N_j^k \tilde{\iota}_j^k = e_j \tilde{\iota}_j = \text{persuasion activity of } G_j, \tag{33}$$

where e_j is a global factor describing the *strength of individual persuasion activities* and $\tilde{\iota}_j$ takes into account the different persuasive power of members with different status.

Therefore we arrive at the following form for the transition rates:

$$\begin{aligned} \nu_I(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C}) &= \nu_0 \exp[m(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C})], \\ \nu_D(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C}) &= \nu_0 e_j \tilde{\iota}_j \exp[m(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C})], \end{aligned} \tag{34}$$

which are combined in the formula for the total transition rate (see Eq. (17)):

$$\begin{aligned} \nu(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C}) &= \nu_0 (1+e_j\tilde{\iota}_j) \exp[m(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C})] \\ &= \nu_0 \exp[m'(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C})] \end{aligned} \tag{35}$$

with the *effective conditional motivation potential*:

$$\begin{aligned} m'(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C}) &= m(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C}) + \ln(1+e_j\tilde{\iota}_j) \\ &= u_j^k(\mathbf{C}_{ji}^{kh}) - u_i^h(\mathbf{C}) - t_{ji}^{kh} + \ln(1+e_j\tilde{\iota}_j). \end{aligned} \tag{36}$$

The *global rate* ν_0 calibrates the transition speed, i.e. the time scale on which the whole transition process will take place.

The following transformations are of purely mathematical nature. They lead to equivalent forms of the total transition rate. However, owing to the postulated exponential form of the transition rate in terms of the effective conditional motivation potential, this transformation leads to additional interpretations which appeal to the intuition and give a further justification of the postulated form of $\nu(\ldots|\ldots)$. In these formal considerations we omit for simplicity the arguments $\mathbf{C}_{ji}^{kh}$ and $\mathbf{C}$ of $m'(jk; \mathbf{C}_{ji}^{kh}|ih; \mathbf{C})$.

At first we decompose the effective conditional motivation potential into a *symmetrical part* m_s' and an *antisymmetrical part* m_a':

$$m'(jk|ih) = m_s'(jk|ih) + m_a'(jk|ih) \tag{37}$$

with

$$\begin{aligned} m_s'(jk|ih) &= m_s'(ih|jk) = \tfrac{1}{2}\{m'(jk|ih) + m'(ih|jk)\}, \\ m_a'(jk|ih) &= -m_a'(ih|jk) = \tfrac{1}{2}\{m'(jk|ih) - m'(ih|jk)\}. \end{aligned} \tag{38}$$

Inserting Eq. (36) and the explicit form of t_{ji}^{kh}, Eq. (24), one obtains more detailed expressions:

$$m_s'(jk|ih) = -\tfrac{1}{2}(a_j^k + a_i^h + l_j^k + l_i^h) + \tfrac{1}{2}\ln[1+e_j\tilde{\iota}_j)(1+e_i\tilde{\iota}_i)] = -\ln d_{ji}^{kh} \tag{39}$$

with

$$d_{ji}^{kh} = d_{ij}^{hk} = \frac{\exp\{\frac{1}{2}(a_j^k + a_i^h + l_j^k + l_i^h)\}}{\sqrt{(1+e_j\tilde{\iota}_j)(1+e_i\tilde{\iota}_i)}} \tag{40}$$

and

$$\begin{aligned} m_a'(jk|ih) &= u_j^k - u_i^h - \tfrac{1}{2}[(a_j^k - l_j^k) - (a_i^h - l_i^h)] \\ &\quad + \tfrac{1}{2}\ln(1+e_j\tilde{\iota}_j) - \tfrac{1}{2}\ln(1+e_i\tilde{\iota}_i) \\ &= v(jk) - v(ih) \end{aligned} \tag{41}$$

with

$$\begin{aligned} v(jk) &= u_j^k - \tfrac{1}{2}(a_j^k - l_j^k) + \tfrac{1}{2}\ln(1+e_j\tilde{\iota}_j), \\ v(ih) &= u_i^h - \tfrac{1}{2}(a_i^h - l_i^h) + \tfrac{1}{2}\ln(1+e_i\tilde{\iota}_i). \end{aligned} \tag{42}$$

If Eq. (37) together with Eqs. (39) and (41) are inserted into (35), one will obtain an alternative though equivalent form of the transition rate:

$$\nu(jk|ih) = \nu_0 \exp[m'(jk|ih)] = \nu_0 \frac{\exp[v(jk) - v(ih)]}{d_{ji}^{kh}}, \tag{43}$$

which suggests the interpretation of d_{ji}^{kh} as an *effective sociological distance* between states (ih) and (jk), and of $v(jk)$, $v(ih)$ as *effective attractivenesses* of the states (jk) and (ih), respectively.

The effective attractiveness $v(ih)$, and correspondingly $v(jk)$, can be further decomposed into positive and negative terms by splitting up u_i^h according to Eq. (21):

$$v(ih) = v_+(ih) - v_-(ih),$$

with

$$v_+(ih)=b_i^h+\tfrac{1}{2}l_i^h+\tfrac{1}{2}\ln(1+e_i\bar{\iota}_i),$$

$$v_-(ih)=c_i^h+\tfrac{1}{2}a_i^h, \qquad (44)$$

where $v_+(ih)$ and $v_-(ih)$ are denoted as *effective pull towards* (ih) and *effective push away from* (ih), respectively. Inserting (44) in (43) yields another equivalent form of the transition rate:

$$v(jk|ih)=\frac{v_0}{d_{ji}^{kh}}\frac{\exp[v_+(jk)]\exp[v_-(ih)]}{\exp[v_-(jk)]\exp[v_+(ih)]}, \qquad (45)$$

which allows an interpretation of the transition rate in terms of "distance", "pulling", and "pushing" factors. Indeed, according to formula (45), the transition rate from the origin state (ih) to the destination state (jk) is large (small)

- for large (small) terms $v_+(jk)$ pulling towards (jk)
- for small (large) terms $v_-(jk)$ pushing away from (jk)
- for large (small) terms $v_-(ih)$ pushing away from (ih)
- for small (large) terms $v_+(ih)$ pulling towards (ih)
- for small (large) effective distances d_{ji}^{kh} between (ih) and (jk).

(d) Growth Rate and Saturation Rate of the Solidarity

Having discussed the elementary "dynamics-generating" quantities of the personal variables, namely the transition rates, we must now also consider the elementary quantities which produce the dynamics of the solidarity variables.

In Section 2 we have already indicated that two counteractive "forces" are at work producing the dynamics of S_i, namely a *growth rate* which is mainly due to the collective activities of the members of G_i to enhance their solidarity, and a *saturation rate* limiting the unrestricted growth due to frustration effects and a limited receptivity for group ideology and objectives among the members.

To formulate these effects in a quantitative form we first introduce an *activity rate* for the growth of solidarity:

$$\alpha_i'=\alpha_i'(\mathrm{C}_i)=(\alpha_{0i}+\alpha_{1i}N_i)(1-S_i)=\alpha_i(N_i)(1-S_i). \qquad (46)$$

The activity rate contains an "absolute" term α_{0i} comprising solidarity creating (perhaps ideological) processes which enhance the sentiments of identity within G_i *independently* of the size of the group, and a term $\alpha_{1i}N_i$ *proportional to the number* of members of G_i. The latter includes all activities of members enhancing the feelings of commonness within G_i. However, as solidarity S_i approaches its maximum value 1 the activity rate is reduced. This is accounted for by the factor $(1-S_i)$. Secondly, we introduce a *saturation rate* which is *counteractive* to the activity rate (and which therefore enters the equation for $S_i(t)$ with a negative sign). The following form seems highly plausible:

$$\sigma_i'=\sigma_i'(\mathrm{C}_i)=\sigma_{0i}S_i+\sigma_{1i}N_i^2S_i=\sigma_i(N_i)S_i. \qquad (47)$$

The term $\sigma_{0i}S_i$ takes into account those saturation effects of S_i which are independent of the size of G_i (for instance the limited receptivity of each member of the group for too much ideology in G_i). The term $\sigma_{1i}S_iN_i^2$ takes into account the saturation of S_i caused by the free-rider effect.

A single individual is stronger tempted to behave in an uncooperative way, the less apparent it will be (i.e. the easier he/she can hide in the crowd of members) and the less necessary it is to cooperate (i.e. the higher the level of solidarity S_i already is). That means the temptation is proportional to N_iS_i. Since this temptation affects every member, the total free-rider effect must be proportional to $N_i \cdot N_iS_i=N_i^2S_i$.

Undoubtedly, one could think of more detailed formulas for $\alpha_i(\mathrm{C})$ and $\sigma_i(\mathrm{C})$ but we consider Eqs. (46) and (47) as the simplest form of these quantities which will lead to a reasonable dynamics of the solidarity variable.

At the end of this chapter we summarize all key-variables, trend functions and trend parameters, their names, brief interpretations, and their defining equations in tables (Tables 5.1–5.3).

3.4. The Dynamic Equations of the Key-Variables

Having supplied the construction elements of the dynamics, namely the transition rates for the collective personal variables and the rates for the solidarity variables, we are equipped with all requisites for setting up the equations of motion for the key-variables.

Indeed, the *equations for the collective personal variables* are nothing but generalized migratory equations (Weidlich and Haag, 1988;

TABLE 5.1
Key-variables

Variables	Name and interpretation	Defining equation
N_0^0	Number of the crowd of individuals belonging to no group	(2)
N_i^h	Number of members of group G_i with status level h	(1)
N_i	Total number of members of group G_i	(1)
N	Total number of individuals in the social system	(2)
$\hat{N}_i^h$	$rnd(N_i^h)=N_i^h$ rounded up to the next higher integer	(49)
$\tilde{N}_i^h$	Saturated number of members of group G_i in status level h	(9)–(11)
N_{max}	Limit of saturated numbers $\tilde{N}_i^h$	(9)–(11)
$\bar{N}_i^h$	Stationary value of the number of members of group G_i in status level h	(57)
n_i^h	Number of personnel (subordinates) per member of status h in group G_i	(6)
S_i	Solidarity variable of group G_i	(14)
$\bar{S}_i$	Stationary value of the solidarity variable of G_i	(57)
$i_i^h=n_i^h+1$	Potential influence in state (ih)	(7)
i_i	Total potential influence of group G_i	(8)
$\tilde{\imath}_i^h$	Saturated potential influence in state (ih)	(12)
$\tilde{\imath}_i$	Total saturated potential influence of group G_i	(13)
$\tilde{\imath}_i S_i$	Effective total influence of group G_i	(26), (30)
$e_i \tilde{\imath}_i$	Persuasion activity of group G_i	(33)
$\mathbf{N}$	Total group configuration = set of numbers $\{N_0^0, \dots N_i^h \dots N_G^{H_G}\}$	(3)
$\mathbf{N}_i$	Partial group configuration of G_i = set of numbers $\{N_i^0, \dots N_i^{H_i}\}$	(3)
$\mathbf{N}_{ji}^{kh}$	Shifted total group configuration after transition of one individual from state (ih) to state (jk)	(4)
$\mathbf{C}$	Total key-variable configuration $\{N_0^0; \mathbf{C}_1 \dots \mathbf{C}_G\}$	(15)
$\mathbf{C}_i$	Partial key-variable configuration of G_i: $\{S_i; \mathbf{N}_i\}$	(15)
$\mathbf{C}_{ji}^{kh}$	Shifted total key-variable configuration = $\{\mathbf{S}, \mathbf{N}_{ji}^{kh}\}$	(15)

Weidlich, 1991). They take the form

$$\frac{dN_j^k}{dt} = \sum_{i,h} \nu(jk; \hat{\mathbf{C}}_{ji}^{kh} | ih; \hat{\mathbf{C}}) \hat{N}_i^h - \sum_{i,h} \nu(ih; \hat{\mathbf{C}}_{ij}^{hk} | jk; \hat{\mathbf{C}}) \hat{N}_j^k,$$

$$\text{for } j=1, \dots, G; \; k=0, \dots, H_j \tag{48}$$

and

$$\frac{dN_0^0}{dt} = \sum_{i,h} \nu(00; \hat{\mathbf{C}}_{0i}^{0h} | ih; \hat{\mathbf{C}}) \hat{N}_i^h - \sum_{i,h} \nu(ih; \hat{\mathbf{C}}_{i0}^{h0} | 00; \hat{\mathbf{C}}) \hat{N}_0^0$$

with

$$\hat{\mathbf{C}} = (\hat{N}_0^0; S_1, \hat{N}_1^1, \dots \hat{N}_1^{H_1}; \dots; S_G, \hat{N}_G^1 \dots \hat{N}_G^{H_G}). \tag{49}$$

TABLE 5.2
Trend-functions

Trend-functions	Name and interpretation	Defining equation
$\nu(jk; \mathbf{C}_{ji}^{kh} \| ih; \mathbf{C})$	Total transition rate from $(ih; \mathbf{C})$ to $(jk; \mathbf{C}_{ij}^{kh})$	(16), (17), (35), (43), (45)
$\nu_D(jk; \mathbf{C}_{ji}^{kh} \| ih; \mathbf{C})$	Partial transition rate induced by *direct* pair interactions	(17), (34)
$\nu_I(jk; \mathbf{C}_{ji}^{kh} \| ih; \mathbf{C})$	Partial transition rate induced by *indirect* interactions	(17), (34)
$m(jk; \mathbf{C}_{ji}^{kh} \| ih; \mathbf{C})$	Conditional motivation potential for transition from $(ih; \mathbf{C})$ to $(jk; \mathbf{C}_{ji}^{kh})$.	(18)–(20)
$m'(jk; \mathbf{C}_{ji}^{kh} \| ih; \mathbf{C})$	Effective conditional motivation potential	(35)–(37)
$m'_s(jk; \mathbf{C}_{ji}^{kh} \| ih; \mathbf{C})$	Symmetrical part of $m'(jk; \mathbf{C}_{ji}^{kh})$	(37)–(39)
$m'_a(jk; \mathbf{C}_{ji}^{kh} \| ih; \mathbf{C})'$	Antisymmetrical part of $m'(jk; \mathbf{C}_{ji}^{kh})$	(37), (38), (41)
$d_{ji}^{kh}(\mathbf{C}_{ji}^{kh}; \mathbf{C})$	Effective sociological distance between states (ih) and (jk)	(40)
$v(ih; \mathbf{C})$	Effective attractiveness of state (ih)	(41), (42)
$v_+(ih; \mathbf{C})$	Effective pull towards state (ih)	(44)
$v_-(ih; \mathbf{C})$	Effective push away from state (ih)	(44)
$u_i^h(\mathbf{C})$	Net utility of state (ih)	(20), (21), (31)
$b_i^h(\mathbf{C})$	Benefits of state (ih)	(21), (22)
$p_i^h(\mathbf{C})$	Payoff of state (ih)	(22), (26)
$f_i(\mathbf{C})$	Faith confirmation for members of G_i	(22), (30)
$c_i^h(\mathbf{C})$	Costs of state (ih)	(21), (23)
$o_i^h(\mathbf{C})$	Obligations in state (ih)	(23), (25)
$q_i(\mathbf{C})$	Contributions of members of group G_i	(23)
$t_{ji}^{kh}(\mathbf{C})$	Transaction costs of transition from state (ih) to state (jk)	(20), (24)
$l_i^h(\mathbf{C})$	Leaving costs when leaving state (ih)	(24)
$a_j^k(\mathbf{C})$	Admission costs when entering state (jk)	(24)
$\alpha'_j(\mathbf{C})$	Activity rate for growth of solidarity S_i	(46)
$\sigma'_i(\mathbf{C})$	Saturation rate for saturation of solidarity S_i	(47)

The values $\hat{N}_i^h = rnd(N_i^h)$ are N_i^h, rounded to the nearest integer. The sums on the right-hand side (r.h.s.) extend over the states $(ih)=(00)$ and (ih) with $i=1, \dots, G$ and $h=0, 1, \dots, H_i$.

The mathematical meaning of Eq. (48) is easily comprehensible: The *first term on the r.h.s.* describes the number of individuals arriving per unit of time in state (jk) and coming from one of the states (ih) by performing the transition $(ih) \rightarrow (jk)$. (Note that the states (ih) include also the state (00) of people not involved in any group, i.e. of the crowd

TABLE 5.3
Trend-parameters

Trend-parameters	Name and interpretation	Defining equation
ν_0	Global rate calibrating the transition speed	(34)
e_j	Strength of individual persuasion activity	(33)
d_i	Exponent of obligations in group G_i	(25)
r_i^h	Share coefficient in the payoff expression	(26), (27)
r_i	reward coefficient	(27)
g_0, g_1	Coefficients calibrating the absolute and influence dependent part of the payoff	(26)
o_i	Global factor calibrating the obligations in G_i	(25)
w_{ij}	Faith feedback coefficients calibrating the faith confirmation	(30)
α_{0i}, α_{1i}	Rate coefficients determining the absolute and member dependent part of the activity rate $\alpha_i'(\mathbf{C})$	(46)
σ_{0i}, σ_{1i}	Rate coefficients determining the natural and member dependent part of the saturation rate $\sigma_i'(\mathbf{C})$	(47)

of individuals.) This term leads to an increase of N_j^k with time. The *second term* on the r.h.s. describes the number of individuals leaving state (jk) and performing transitions into any of the states (ih) (including (00)). This term leads to a decrease of N_j^k with time. Hence the change of N_j^k with time comes about by the counter-active effects of transitions into (jk) and transitions out of (jk). The analogous holds for Eq. (49), the evolution equation for the non-members. From (48) and (49) there follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(N_0^0+\sum_{j=1}^{G}\sum_{k=0}^{H_j}N_j^k\right)=\sum_{jk}\sum_{ih}\nu(jk;\hat{\mathbf{C}}_{ji}^{kh}|ih;\hat{\mathbf{C}})\hat{N}_i^h$$

$$-\sum_{jk}\sum_{ih}\nu(ih;\hat{\mathbf{C}}_{ij}^{hk}|jk;\hat{\mathbf{C}})\hat{N}_j^k=0, \qquad (50)$$

i.e. the total number N of individuals remains constant in time. Therefore the "conservation law" expressed in Eq. (2) is compatible with Eqs. (48) and (49).

The quantities $N_0^0(t)$, $N_j^k(t)$ take on continuous values since they are mean-values for an ensemble of comparable systems of groups. However, for small groups some of their values can become very small (even between 0 and 1) so that $\mathbf{N}_{kh}^{ji}$ could formally contain non-allowed

negative values. Therefore it is reasonable to insert (at evaluating $\mathrm{d}N_j^k/\mathrm{d}t$ and $\mathrm{d}S_i/\mathrm{d}t$) the values $\hat{N}_i^h = rnd(N_i^h)$ (and $\hat{N}_j^k = rnd(N_j^k)$) which are rounded to the nearest integers. This accounts for the discreteness of the number of individuals. For sufficiently large numbers $N_i^h, N_j^k \gg 1$ the rounding procedure plays no significant role. The equations for the solidarity variables can also be obtained in straight manner: On the one hand, the relative rate of change of a variable $S_i(t)$ is defined by

$$\frac{\mathrm{d}S_i/\mathrm{d}t}{S_i(t)} = \text{relative rate of change of } S_i(t). \qquad (51)$$

On the other hand, it is our proposition that this relative rate of change is composed of two counteractive terms, namely the activity rate $\alpha_i'(\hat{\mathbf{C}})$ and the saturation rate $\sigma_i'(\hat{\mathbf{C}})$.

$$\frac{\mathrm{d}S_i/\mathrm{d}t}{S_i(t)} = \alpha_i'(\hat{\mathbf{C}}) - \sigma_i'(\hat{\mathbf{C}}). \qquad (52)$$

Rewriting (52) and inserting Eqs. (46) and (47) one obtains:

$$\frac{\mathrm{d}S_i}{\mathrm{d}t} = \alpha_i(\hat{N}_i)S_i - (\alpha_i(\hat{N}_i)+\sigma_i(\hat{N}_i))S_i^2 \quad \text{for } i=1,2,\ldots,G. \qquad (53)$$

This dynamic equation for the solidarity variable S_i is a kind of logistic equation (see Pearl, 1924; Verhulst, 1845). However, the difference to the original logistic equation is that the coefficients $\alpha_i(\hat{N}_i)$ and $(\alpha_i(\hat{N}_i)+\sigma_i(\hat{N}_i))$ are not constants but functions of $\hat{N}_i$, the number of members of group G_i, which is a variable itself.

The $\sum_{i=1}^{G}(H_i+1)+1+G = (\sum_{i=1}^{G}H_i+2G+1)$ Eqs. (48), (49), and (53) for the G interacting groups G_i, each of them consisting of (H_i+1) status levels, and the crowd of non-members establish *the mathematical form of our group-dynamic model*. They become fully explicit by inserting the form (35) or (43) of the transition rates and by expressing them via (36) with (31) in terms of the key-variables. It is obvious that this set of equations is a coupled non-linear autonomous system of first-order differential equations in time.

3.5. Initial Conditions and Stationary Solutions

In this section we will first make a general remark about the *problem of the initial state* of the evolution of groups in view of our equations; secondly we will search for the structure of the *stationary solutions* of the model equations in important special cases where the generalized migratory equations fulfil the condition of "detailed balance".

The "natural" initial state for a system of emerging groups is the state where groups do not yet exist. This state can be characterized in our system by

$$N_0^0 = N, \quad N_i^h = 0, \quad S_i = 0, \quad i = 1, \ldots, G, \ h = 0, \ldots, H_i \tag{54}$$

and it will turn out to be a metastable state (i.e. not a fully stable state) if our mean-value equations are formally applied to that extreme case. In fact, we obtain for this initial state

$$v(00) = 0 \quad \text{and} \quad v(jk) < 0 \ \text{for} \ (jk) \neq (00) \tag{55}$$

because for the state (54) we have (putting $a_j^k = l_j^k$)

$$v(jk) = u_j^k \quad \text{with} \ u_j^k = (p_j^k + f_j) - (o_j^k + q_j) < 0 \tag{56}$$

since the payoff and faith confirmation are equal to 0 for the not yet existing groups but obligations and contributions do already exist for anyone wanting to initiate a group. Therefore the transition rate from the non-member state (00) to any state $(jk) \neq (00)$ is, according to (43), very small since it is more attractive to stay in the non-member state than to take the burden of obligations and contributions connected with the foundation of a group.

However, mean-value equations like those of our proposed model are not really applicable to a state like (54) with vanishing or extremely small numbers of members in all groups. In such cases statistical fluctuations play a dominant role so that a fully probabilistic formulation (e.g. in terms of the master equation (see Weidlich and Haag, 1983; Weidlich, 1991; 1994; Helbing, 1995)) for a theory of very small emerging groups is indispensable. Such a stochastic theory of the emergence of new groups would be a highly interesting topic but it is beyond the scope of the approach presented here.

A quite different situation is given if we investigate the structure of possibly existing *stationary solutions* of the system equations with time-independent key-variables:

$$\bar{\mathbf{C}} = \{\bar{N}_0^0; \bar{S}_1, \bar{N}_1^0 \ldots \bar{N}_1^{H_1}; \ldots; \bar{S}_i, \bar{N}_i^0 \ldots \bar{N}_i^{H_i}; \ldots; \bar{S}_G, \bar{N}_G^0 \ldots \bar{N}_G^{H_G}\}. \tag{57}$$

We can expect that in such states for one or more groups the numbers of members $\bar{N}_i^h$ will be large compared to 1 so that the mean-value equations can safely be applied.

The stationary values (57) of the key-variables must evidently fulfil the stationary equations derived from Eqs. (48), (49) and (53) by

putting the time derivatives equal to zero. This yields:

$$0 = \sum_{ih} \frac{\nu_0}{d_{ji}^{kh}(\bar{\mathbf{C}})} \{\exp[v(jk, \bar{\mathbf{C}}) - v(ih, \bar{\mathbf{C}})]\bar{N}_i^h - \exp[v(ih, \bar{\mathbf{C}}) - v(jk, \bar{\mathbf{C}})]\bar{N}_j^k\} \tag{58}$$

and

$$0 = \sum_{ih} \frac{\nu_0}{d_{0i}^{0h}(\bar{\mathbf{C}})} \{\exp[v(00, \bar{\mathbf{C}}) - v(ih, \bar{\mathbf{C}})]\bar{N}_i^h - \exp[v(ih, \bar{\mathbf{C}}) - v(00, \bar{\mathbf{C}})]\bar{N}_0^0\}, \tag{59}$$

where the sums extend over (ih) with $i = 1, \ldots, G$, $h = 0, \ldots, H_i$ and $(ih) = (00)$, as well as

$$0 = \alpha_i(\bar{N}_i)\bar{S}_i - (\alpha_i(\bar{N}_i) + \sigma_i(\bar{N}_i))\bar{S}_i^2. \tag{60}$$

Here we have already used the explicit form (43) of the transition rates, and have neglected the shift $\bar{\mathbf{C}} \to \bar{\mathbf{C}}_{ji}^{kh}$ because the $\bar{N}_j^k, \bar{N}_i^h$ were assumed large compared to 1.

The stationary solidarity equations (60) can easily be solved with the result

$$\bar{S}_i = \frac{\alpha_i(\bar{N}_i)}{\alpha_i(\bar{N}_i) + \sigma_i(\bar{N}_i)} < 1, \tag{61}$$

which shows that $\bar{S}_i$ is a function of $\bar{N}_i$ and is always in the required domain (14) between 0 and 1 (see Figure 5.2).

The result (61) can be inserted in (58) and (59) to obtain a set of equations depending on the single components $\{\bar{N}_0^0, \bar{N}_1^0, \ldots, \bar{N}_G^{H_G}\}$ of the stationary group configuration $\bar{\mathbf{N}} = \{\bar{N}_0^0, \bar{\mathbf{N}}_1 \ldots \bar{\mathbf{N}}_G\}$ only (instead of depending on both $\bar{\mathbf{N}}$ and $\bar{\mathbf{S}}$). We formulate this by substituting the following terms in (58), (59):

$$v(jk, \bar{\mathbf{C}}) \to v(jk, \bar{\mathbf{N}}), \quad v(ih, \bar{\mathbf{C}}) \to v(ih, \bar{\mathbf{N}}) \quad \text{and} \quad d_{ji}^{kh}(\bar{\mathbf{C}}) \to d_{ji}^{kh}(\bar{\mathbf{N}}). \tag{62}$$

Before turning to the solution of Eqs. (58) and (59) we have to specify the value of the effective attraction $v(00, \bar{\mathbf{N}})$ of the set of non-members which has special properties. According to the general formula (42),

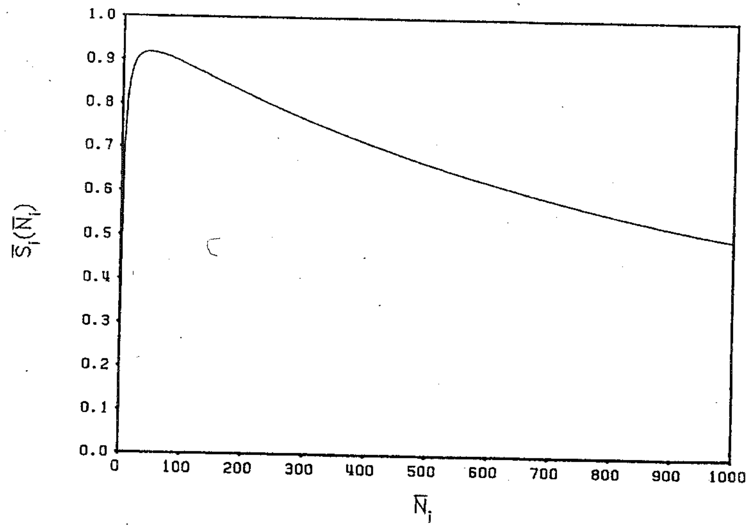

FIGURE 5.2. Stationary values $\bar{S}_i$ of solidarity in a group G_i as function of the stationary total number $\bar{N}_i$ of members of G_i according to eq. (60). The trend-parameters are chosen as $\alpha_{0i}=0.1$, $\alpha_{1i}=0.5$, $\sigma_{0i}=1.0$, $\sigma_{1i}=0.0005$.

$v(00,\bar{\mathbf{N}})$ should have the form

$$v(00,\bar{\mathbf{N}})=u_0^0(\bar{\mathbf{N}})-\frac{1}{2}(a_0^0-l_0^0)+\frac{1}{2}\ln(1+e_0\tilde{\imath}_0)$$

with

$$u_0^0(\bar{\mathbf{N}})=(p_0^0+f_0)-(o_0^0+q_0). \tag{63}$$

Since in the crowd of non-members there exist neither obligations ($o_0^0=0$) and contributions ($q_0=0$) nor payoff ($p_0^0=0$) and faith confirmation ($f_0=0$) (at least in the special case $w_{0j}=0$), we obtain $u_0^0(\bar{\mathbf{N}})=0$. Furthermore, no admission and leaving costs exist ($a_0^0=l_0^0=0$) and no persuasion activities to enter the crowd of non-members are developed ($e_0=0$). Therefore one comes to the conclusion that

$$v(00,\bar{\mathbf{N}})=0. \tag{64}$$

For solving Eqs. (58) and (59) let us consider the much simpler set of equations

$$\bar{N}_i^h=C\exp[2v(ih,\bar{\mathbf{N}})] \tag{65}$$

where $i=1,\ldots,G$, $h=0,\ldots,H_i$, or $(ih)=(00)$.

If the system of transcendental equations (65) for $\bar{\mathbf{N}}=\{\bar{N}_0^0,\bar{\mathbf{N}}_1,\ldots\bar{\mathbf{N}}_G\}$ has one or more than one solution at all, these solutions are simultaneously solutions of Eqs. (58) and (59), i.e. they are stationary group configurations! Indeed, inserting (65) in (58) and (59) one obtains:

$$\begin{aligned}&\{\exp[v(jk,\bar{\mathbf{N}})-v(ih,\bar{\mathbf{N}})]\bar{N}_i^h-\exp[v(ih,\bar{\mathbf{N}})-v(jk,\bar{\mathbf{N}})]\bar{N}_j^k\}\\&\quad=C\{\exp[v(jk,\bar{\mathbf{N}})+v(ih,\bar{\mathbf{N}})]-\exp[v(ih,\bar{\mathbf{N}})+v(jk,\bar{\mathbf{N}})]\}=0.\end{aligned} \tag{66}$$

This means that the brackets in the sums of (58) and (59) vanish individually so that (58) and (59) are fulfilled.

The meaning of Eq. (66) will become clear if one multiplies (66) by $v_0/d_{ji}^{kh}(\bar{\mathbf{N}})$ and re-inserts the expression (43) of the transition rate. Then (66) reads

$$\nu(jk|ih;\bar{\mathbf{N}})\bar{N}_i^h=\nu(ih|jk;\bar{\mathbf{N}})\bar{N}_j^k. \tag{67}$$

This relation is called *detailed balance*. It means that the stationary flow of individuals (per unit of time) from state (ih) to state (jk) is the same as the stationary flow from state (jk) to state (ih) *for each pair of states* (ih),(jk). If (65) has no solutions, there may still exist solutions of (58) and (59) because, for satisfying the latter equations, detailed balance is only a *sufficient* but not a *necessary* condition. Equations (58) and (59) require only that for each state (jk) the *total* inflow is equal to the *total* outflow.

Trying to solve Equations (65) one obtains with (64) in any case:

$$\bar{N}_0^0=C\exp[0]=C; \tag{68}$$

where the constant C has to be determined by the normalization condition (see (1)):

$$\bar{N}_0^0+\sum_{i=1}^{G}\sum_{h=0}^{H_i}\bar{N}_i^h=C\left(1+\sum_{i=1}^{G}\sum_{h=0}^{H_i}\exp[2v(ih,\bar{\mathbf{N}})]\right)=N. \tag{69}$$

Apart from this simple constraint the complexity of the set of equations (65) depends on the form of $v(ih,\bar{\mathbf{N}})$ as a function of the stationary group configuration $\bar{\mathbf{N}}$.

The solution of (65) is considerably simplified if $v(ih,\bar{\mathbf{N}})$ depends only on $\bar{\mathbf{N}}_i$, i.e. on the configuration of group G_i. The meaning of this simplification is that the effective attraction $v(ih,\bar{\mathbf{N}}_i)$ depends only on the size and compositionof G_i but not on interfering activities of other groups. Under this condition $v(ih, \bar{\mathbf{N}}) = v(ih, \bar{\mathbf{N}}_i)$ Equations (65) split up in G independent sets, one for each configuration $\bar{\mathbf{N}}_i = \{N_i^0, \ldots, N_i^G\}$, which are connected only by the normalization condition (69).

4. SIMULATION OF SELECTED SCENARIOS

In the previous section we have seen that the *stationary solutions* of the dynamic equations (48), (49) and (53) still have a relatively simple structure if the condition of detailed balance (67) holds. In this case the stationary solutions fulfil Eqs (65).

However, the *time-dependent solutions* of the dynamic equations do not have a simple, analytically tractable solution. They can only be obtained numerically by computer calculations.

These solutions represent an *immense manifold of possible evolutionary scenarios*: To each choice of initial conditions for the key-variables together with the calibrated trend-parameters from Table 5.3 there belongs exactly one out of an infinite number of possible scenarios.

Even if we exclude cases describing the rise of new groups (because of the problem of initial fluctuations for small numbers N_i^h), several characteristic scenarios are expected already in the simple case of two initial groups. For instance:

(a) a stable stationary state of co-existence of both groups could evolve,
(b) one group could score off or push out the other, for instance, by providing a better benefits–costs ratio,
(c) one group could wear down the other by destroying its internal faith with negative interference.

At investigating the structure of such cases one can make use of one advantage of the scenario technique: Whereas empirical situations always contain a *complex superposition* of (too) many factors of influence, the construction of fictitious scenarios can take place by *selectively* and *successively* "switching on" one trend parameter after another. Then, by studying the corresponding solutions of the equations, the effect of each of these trend parameters on the dynamic process can be distinguished and isolated more easily.

Using this circumstance, our procedure will consist in making a small excursion into the "vast forest of possible scenarios". (Of course the stations on this route do not at all exhaust the manifold of possible

evolutions included in the model equations.) In detail this means

1. choosing the *same structure of the group configuration* in all simulations; it consists of two groups G_1 and G_2, each with three status levels $h = 0, 1, 2$, and the crowd of non-members,
2. choosing in all simulations the *same initial conditions* for the key variables $N_0^0; N_1^0, N_1^1, N_1^2, S_1; N_2^0, N_2^1, N_2^2, S_2$,
3. keeping the *same constant values* for *most* of the trend parameters in all simulations,
4. selecting just a *small group* of trend-parameters the values of which are *subject to changes* when passing from one scenario to another, but
5. changing the value of only one *trend-parameter* (which will be underlined) when passing on the "route of scenarios" from *one* scenario to the next.

For each of the seven "stations" on this route, we will present one corresponding figure including four illustrations which depict, as functions of time,

(a) the total numbers of individuals of the crowd and of the two groups: $N_0(t) = N_0^0(t)$(—); $N_1(t) = \sum_{h=0}^{2} N_1^h(t)$(---); $N_2(t) = \sum_{h=0}^{2} N_2^h(t)$ (-·-·-),
(b) the solidarity variables $S_1(t)$(---) and $S_2(t)$ (-·-·-),
(c) the occupation numbers of the hierarchy levels of group G_1: $N_1^0(t)$(—), $N_1^1(t)$(---), $N_1^2(t)$(-·-·-),
(d) the occupation numbers of the hierarchy levels of group G_2: $N_2^0(t)$(—), $N_2^1(t)$(---), $N_2^2(t)$(-·-·-).

Each scenario will be accompanied by a short interpretation stressing particularly the parameter changes in comparison to the respective previous scenario.

The initial conditions for the key-variables are chosen as follows:

$$\begin{aligned} &N_0^0(0) = N_0(0) = 500, \quad N_1(0) = 160, \quad N_2(0) = 90, \\ &S_1(0) = 0.8, \quad S_2(0) = 0.9, \\ &N_1^0(0) = 125, \quad N_1^1(0) = 30, \quad N_1^2(0) = 5, \\ &N_2^0(0) = 75, \quad N_2^1(0) = 15, \quad N_2^2(0) = 0. \end{aligned} \tag{70}$$

At $t = 0$ we begin with already sufficiently large groups in order to avoid the problems connected with emerging groups discussed in Section 3.5 which are beyond the scope of the present article.

TABLE 5.4

$\alpha_{0i}=0.1$; $\alpha_{1i}=0.5$	Partial rates of the activity rate
$\sigma_{0i}=1.0$; $\sigma_{1i}=0.0005$	Partial rates of the saturation rate
$q_i=6.0$	Contributions
$r_1=0.7$	Reward coefficient of group G_1
$d_i=1.0$	Exponents of obligations of groups G_i
$\nu_0=1.0$	Global scaling factor of transition rates
$e_0=0.0$; $e_1=0,0$	Persuasion coefficients of crowd and group G_1
$a_1^h=0.0$	Admission costs of group G_1
$l_i^h=0.0$	Leaving costs
$g_0=0.7$	Coefficients of influence-independent payoff
$w_{01}=w_{11}=w_{21}=w_{22}=0.0$	Faith-feedback coefficients
$o_i=3.0$	Calibration factor of obligations
$N_{\max}=100$	Saturation level

Table 5.4 lists the values of all those trend parameters which are kept constant throughout all scenario simulations. The index $i=1,2$ refers to both groups G_1 and G_2.

The values of all trend parameters not listed above may change from one scenario to another. The variable trend parameters are: g_1=coefficient of influence-dependent payoff, e_2=persuasion coefficient of group G_2, a_2^h=admission costs of group G_2, w_{12}, w_{02}=faith feedback coefficients, r_2=reward coefficient of group G_2. The values of these trend parameters are defined separately for each scenario.

We begin with

Scenario 1

Choice of the variable trend parameters:

$$g_1=0.0,\quad e_2=0.0,\quad a_2^h=0.0,\quad w_{12}=0.0,\quad w_{02}=0.0,\quad r_2=0.7.$$

Interpretation. In this case all trend parameters referring to groups G_1 and G_2 are equal. No group exerts influence on the internal faith of the other group (because all faith influence coefficients are zero). The influence-independent payoff coefficient is small ($g_0=0.7$, corresponding to 30% administrative losses) and the influence-dependent payoff coefficient is zero ($g_1=0.0$). Therefore membership in one of the groups is not really (materially or immaterially) profitable and remunerative. As expected, the result is (see Figure 5.3): The total number of members of both groups decreases; the groups decay and their members join the crowd of non-members. The originally rather high solidarity levels (given by the initial conditions) increase as long as there are still enough members to pursue their activities. Afterwards,

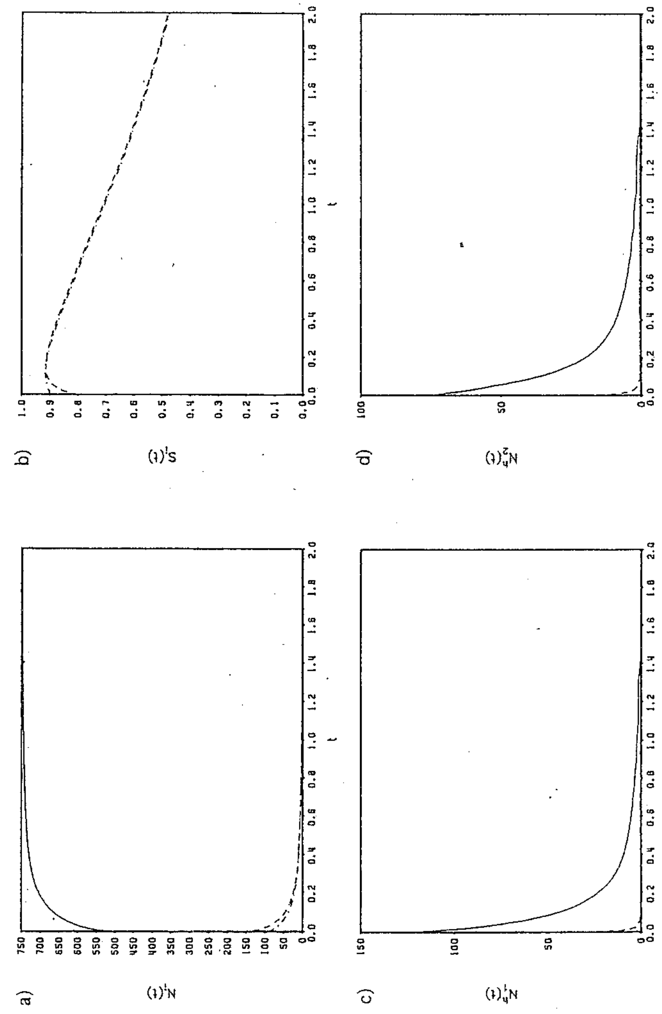

FIGURE 5.3. Belongs to Scenario 1. Its components show (a) the fast decay of the total numbers of group members $N_1(t), N_2(t)$, (b) the slow decay of solidarities $S_1(t), S_2(t)$, (c) and (d) the fast decay of the occupation numbers $N_1^h(t), N_2^h(t)$ of the status levels of both groups G_1 and G_2 since group membership is not profitable.

with the decay setting in, the solidarity level decreases, too. The number of leading members (with status $h=1,2$) decreases even faster than the number of nominal members because there is no sufficient reward for their higher obligations. Consequently the higher obligations induce an earlier leaving of the group.

We proceed to

Scenario 2

with the following choice of the variable trend parameters:

$$\underline{g_1=0.004}, \quad e_2=0.0, \quad a_2^h=0.0, \quad w_{12}=0.0, \quad w_{02}=0.0, \quad r_2=0.7.$$

Interpretation. The payoff situation has now changed: For both groups a positive coefficient of the influence-dependent payoff was introduced. Both influences ("power") $\tilde{\imath}_i(\mathbf{N}_i)$ and solidarity S_i now play a positive role leading to a higher (material and immaterial) payoff. As a consequence the positive term p_i^h in u_i^h now prevails, i.e. membership in groups G_1 and G_2 is now profitable and more individuals of the crowd enter the groups. This can be seen in Figure 5.4. It is also evident that stationary occupation numbers are approached independently of the initial values. Also, for both groups, the solidarity levels approach the same high stationary values, and the distribution of members over the status levels reaches the same stationary ratios. This scenario therefore represents the typical case of two co-existing groups without asymmetry and without interference between them.

We now pass on our route to

Scenario 3

with the following choice of the variable trend parameters:

$$g_1=0.004, \quad \underline{e_2=0.01}, \quad a_2^h=0.0, \quad w_{12}=0.0, \quad w_{02}=0.0, \quad r_2=0.7.$$

Interpretation. This provides an asymmetric behavior of groups G_1 and G_2 since the members of G_2 (and only of G_2) now canvass new members for their group via direct (pair-)interactions, in particular with the individuals of the crowd. The result can be seen in Figure 5.5 which shows that group G_2 reaches a much higher number of members due to its persuasion activity than group G_1 – at the cost of group G_1 and the crowd. From Figure 5.5 it can also be concluded that this result was *not* due to the indirect influence of solidarity because the higher stationary number of members in G_2 leads even to a lower saturation level of the solidarity in G_2 as compared to the solidarity

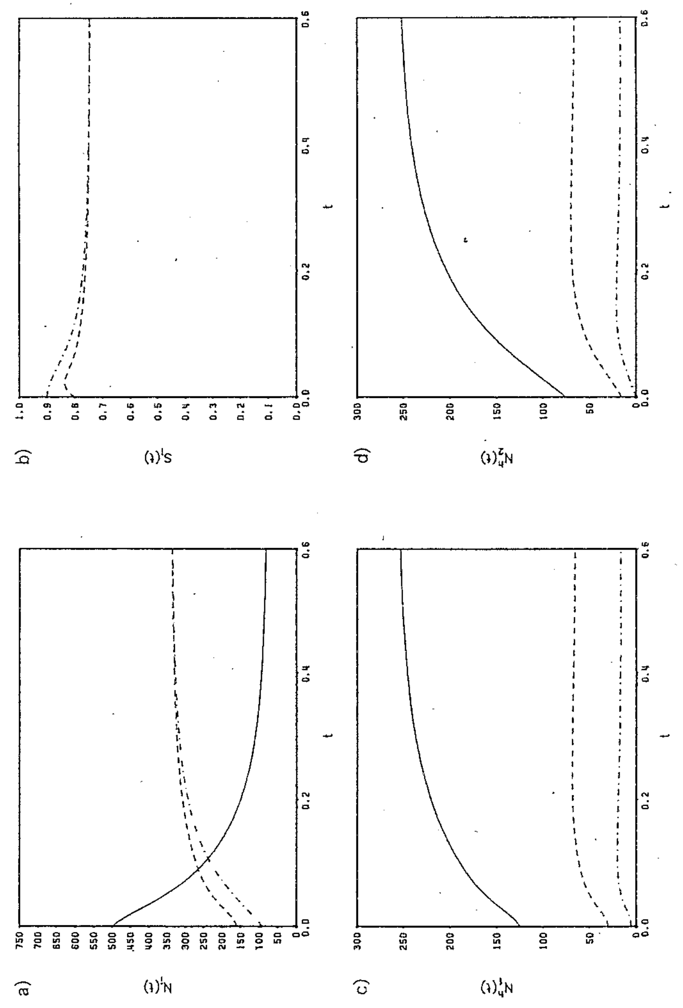

FIGURE 5.4. Belongs to Scenario 2. Its components show (a) the stabilization of $N_1(t)$ and $N_2(t)$ at the same stationary values, (b) the stabilization of $S_1(t)$ and $S_2(t)$ at the same stationary values, (c) and (d) the evolution of the same occupation structure of status levels in groups G_1 and G_2 because the groups apply the same strategy and do not influence each other.

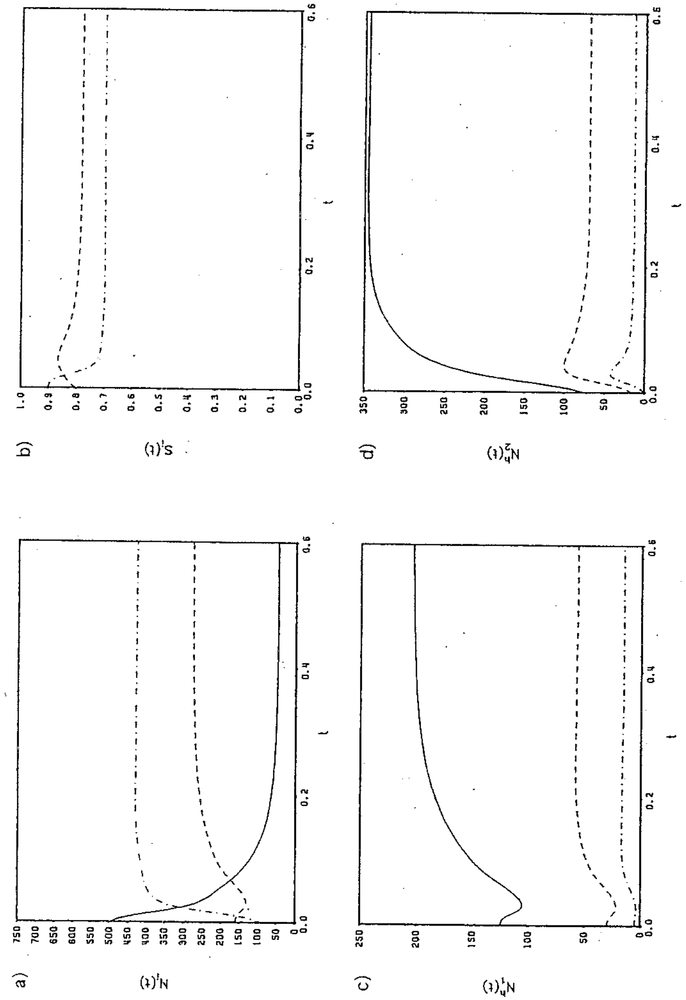

FIGURE 5.5. Belongs to Scenario 3. Its components show (a) the stabilization of $N_1(t)$ and $N_2(t)$ at different stationary values, (b) the stabilization of $S_1(t)$ at different stationary values, (c) and (d) the evolution of different occupation numbers of status levels as a consequence of different total numbers of members in G_1 and G_2. The differences of the stationary levels are due to the canvassing activities of group G_2.

level in G_1. Now the distribution ratio between nominal members and leading members of group G_2 has changed. One observes a relative increase of the number of nominal members due to the increase of the total number of members whereas nothing has changed in this respect in group G_1.

Continuing our journey we arrive at

Scenario 4

with the following choice of the variable trend parameters:

$$g_1=0.004,\quad e_2=0.01,\quad \underline{a_2^h=2.0},\quad w_{12}=0.0,\quad w_{02}=0.0,\quad r_2=0.7.$$

Interpretation. Once more group G_2 is assumed to change its behavior, whereas group G_1 keeps its trend parameters constant. Group G_2 now demands admission fees from all new members. These fees are of course additional costs for the potential new members so that the relative attractiveness of group G_1 has grown. In other words: the greed of group G_2 to make profit of new members was a mistake. This can be seen in Figure 5.6. Group G_1 now keeps a permanent superiority over group G_2 in terms of the approached total number of members. However, the stationary solidarity level in G_2 is slightly higher than in G_1 because the number of members in G_2 is closer to that yielding an optimal stationary solidarity than the corresponding number in G_1 (confer also Figure 5.2). The distribution of members over the status levels in G_1 and G_2 reflects the dependence of their ratio on the respective total number of members.

Proceeding to the next station on the scenario route we arrive at

Scenario 5

with the following choice of the variable trend parameters:

$$g_1=0.004,\quad e_2=0.01,\quad a_2^h=2.0,\quad \underline{w_{12}=-0.003},\quad w_{02}=0.0,\quad r_2=0.7.$$

Interpretation. Again group G_2 is assumed to change its behavior, while G_1 sticks to its trend parameters. Group G_2 is now trying to undermine the faith of group G_1 in its values. This is expressed by the negative faith influence coefficient w_{12}. The strategy is successful because Figure 5.7 shows that now (in spite of the admission fee) group G_2 gains an advantage over G_1 with respect to the total numbers of members. The distributions over status levels in G_1 and G_2 vary depending on the total numbers of members, and also the values of solidarity S_1, S_2 take values corresponding to N_1 and N_2.

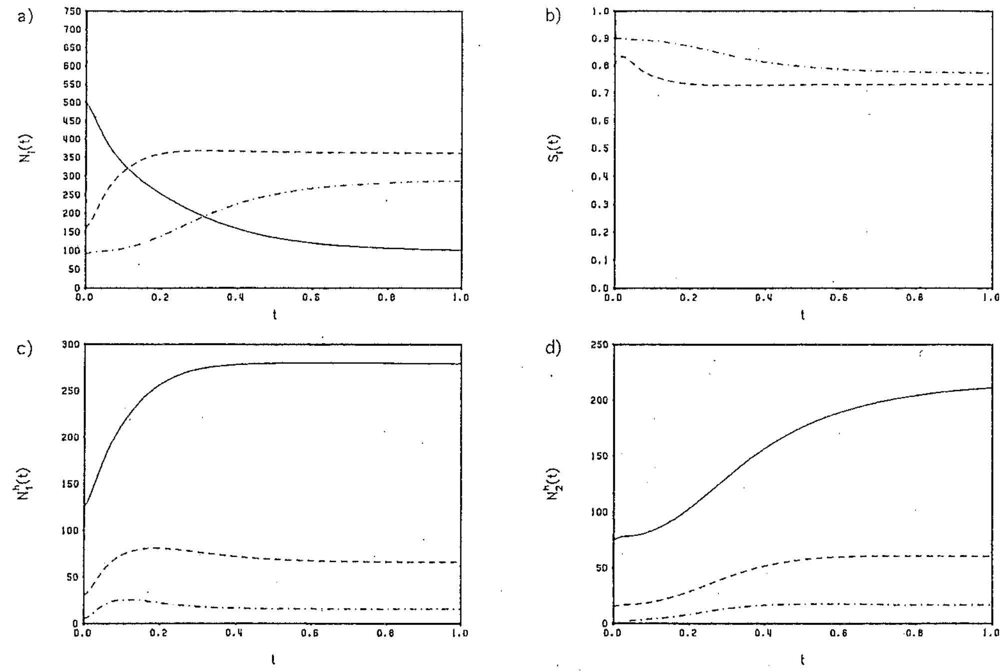

FIGURE 5.6. Belongs to Scenario 4. Its components show (a) that $N_1(t)$ stabilizes at a higher value than $N_2(t)$, (b) that $S_1(t)$ stabilizes at a lower value than $S_2(t)$, (c) and (d) that the occupation numbers in G_1 and G_2 result as a consequence of different total numbers of members in G_1 and G_2. The new stationary levels are due to admission fees demanded by group G_2.

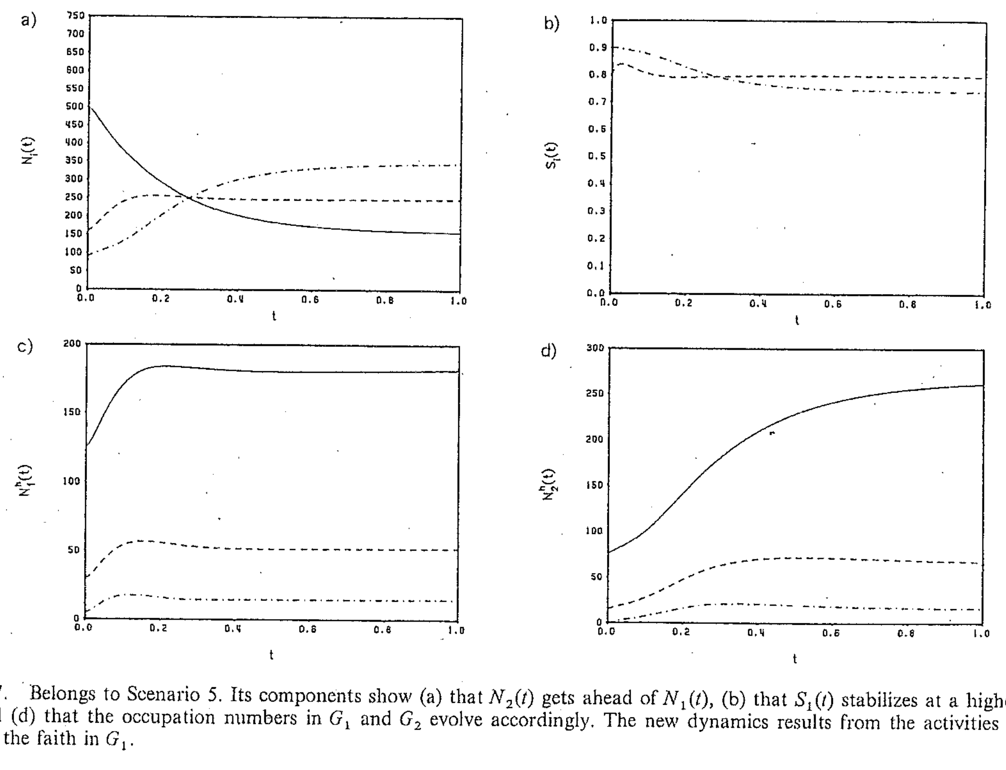

FIGURE 5.7. Belongs to Scenario 5. Its components show (a) that $N_2(t)$ gets ahead of $N_1(t)$, (b) that $S_1(t)$ stabilizes at a higher level than $S_2(t)$, (c) and (d) that the occupation numbers in G_1 and G_2 evolve accordingly. The new dynamics results from the activities of group G_2 undermining the faith in G_1.

Passing to the next station we reach

Scenario 6

with the following choice of the variable trend parameters:

$g_1=0.004,\ e_2=0.01,\ a_2^h=2.0,\ w_{12}=-0.003,\ \underline{w_{02}=0.003},\ r_2=0.7.$

Interpretation. Group G_2 is now assumed to try out an additional trick to disturb group G_1 whereas this group G_1 sticks to its old behavior. G_2 supports and stabilizes the mood of the crowd of individuals (by the positive faith influence coefficient w_{02}) thus prohibiting transitions to G_1 in addition to weaken the faith within G_1. Both influence factors w_{12} and w_{02} have a dramatic effect on the evolution of G_1. One can see in Figure 5.8 that, in spite of an initial advantage and of initial growth of group G_1, the cunning group G_2 catches up and gets ahead of group G_1 which finally breaks down completely. The solidarity within G_1 which exceeds that of G_2 for a long time cannot prevent this breakdown. Finally only G_2 survives having a stabilized status level structure whereas G_1 has decayed completely.

Our journey on the scenario route ends at

Scenario 7

with the following choice of the variable trend parameters:

$g_1=0.004,\ e_2=0.01.\ a_2^h=2.0,\ w_{12}=-0.003,\ w_{02}=0.003,\ \underline{r_2=1.0}.$

Interpretation. Group G_2 which has defeated group G_1 under the trend parameter conditions of the previous scenario is now assumed to introduce another “innovation”, namely a benefit for its leading members by increasing the reward coefficient r_2. However, this “benefit” proves to be lethal for group G_2 and advantageous for group G_1. As one can see in Figure 5.9, the ratio of the numbers of members in the leading hierarchy levels compared to the number of nominal members in G_2 increases substantially after a short time. However, this means that G_2 is quickly becoming “hydrocephalic” and that not much payoff is left for the nominal members. Thus there is no incentive to enter this group G_2 as a simple member. Therefore the fast decline of G_2 is inevitable. The disappearance of the disturbing influences of G_2 on G_1 thereupon leads to a straight evolution of G_1 to a high total number of members and a regular distribution of the members over status levels. The same holds for the solidarity in group G_1 which approaches its stationary value whereas, as expected, the solidarity of G_2 slowly decays.

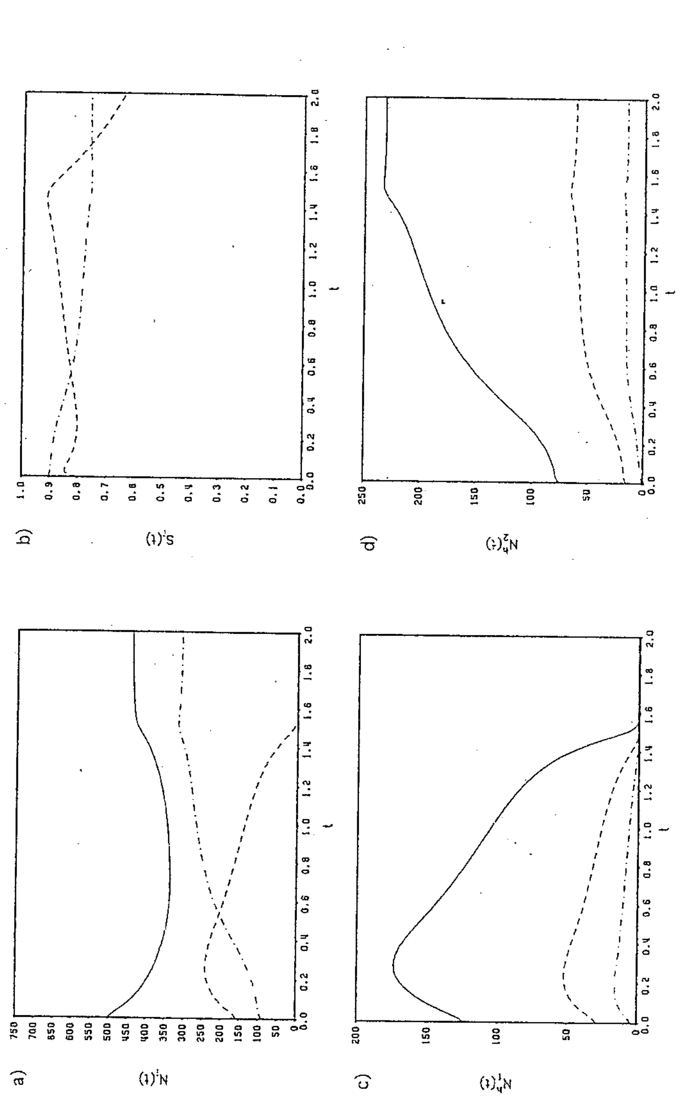

FIGURE 5.8. Belongs to Scenario 6. Its components show (a) the breakdown of $N_1(t)$ and the stabilization of $N_2(t)$, (b) the corresponding evolutions of $S_i(t)$ ending with the final decay of $S_1(t)$ and the stabilization of $S_2(t)$, (c) and (d) the corresponding breakdown of $N_1^h(t)$ and the stabilization of $N_2^h(t)$. The new dynamics results from stabilizing the crowd and from preventing transitions to G_1 by interfering activities of G_2.

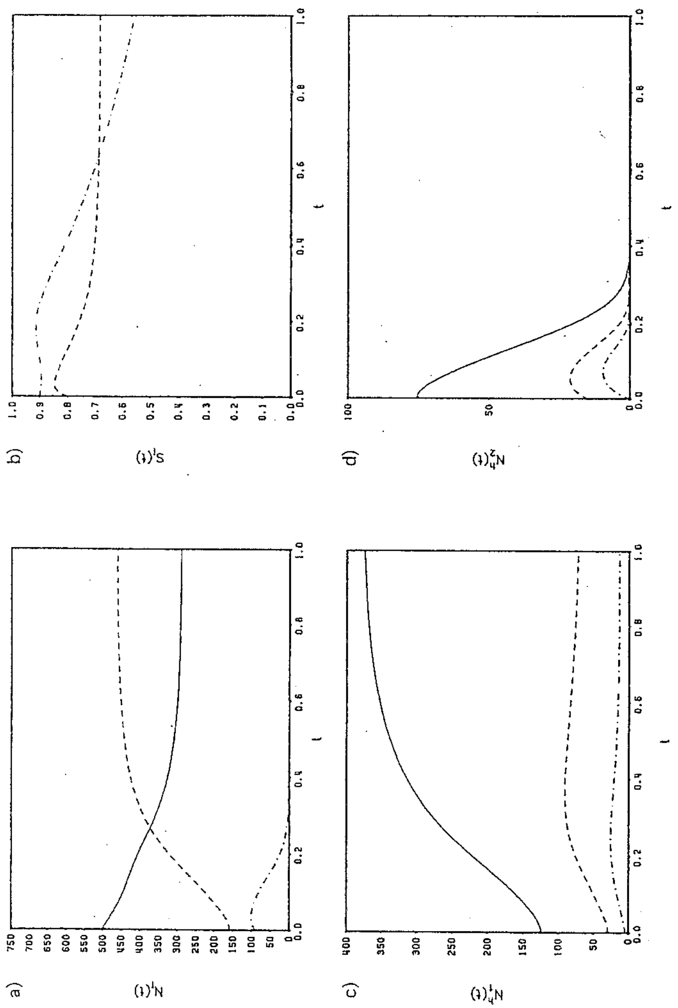

FIGURE 5.9. Belongs to Scenario 7. Its components show (a) the breakdown of $N_2(t)$ and the stabilization of $N_1(t)$, (b) the corresponding evolutions of $S_l(t)$ ending with the final decay of $S_2(t)$ and the stabilization of $S_1(t)$, (c) and (d) the corresponding stabilization of $N_1^h(t)$ and the breakdown of $N_2^h(t)$. The new dynamics results from the introduction of exaggerated rewards for staff members of G_2.

On the way from one scenario to another, passing some possible choices of trend parameters, we have seen that rather different outcomes arise according to the "strategies" (expressed by the trend parameters) chosen by the interacting groups. Taking into account that all trend parameters could in principle also slowly vary with time one becomes aware of the immense complexity of group dynamics even if captured by such simple macroscopic models like ours.

For better understanding of Figures 5.3–5.9 we repeat our drawing conventions. Each figure consists of four illustrations representing

(a) the numbers of individuals $N_0(t)$ of the crowd (—), the total number $N_1(t)$ of members of G_1(- - -), and the total number $N_2(t)$ of members of G_2(-·-·-·-),
(b) the solidarity level $S_1(t)$ of G_1(- - -), the solidarity level $S_2(t)$ of G(-·-·-·-),
(c) the occupation numbers N_1^h of the status levels h in G_1: $N_1^0(t)$(—), $N_1^1(t)$(- - -), $N_1^2(t)$(-·-·-·-),
(d) the occupation numbers N_2^h of the status levels h in G_2: $N_2^0(t)$(—), $N_2^1(t)$(- - -), $N_2^2(t)$(-·-·-·-).

5. THE MODEL IN THE LIGHT OF SELECTED APPROACHES TO GROUP FORMATION IN THE LITERATURE

We will now consider a small part of the comprehensive literature about group formation under the very restrictive point of view of elucidating the accomplishments and limitations of our model. Even so we need some ordering principles serving as guidelines on our way of evaluating the literature. These guidelines will facilitate the positioning of the achievements of some authors in view of their meaning for our modelling procedure. Some of the ordering principles have already been indicated in the introduction and are repeated here.

(a) Reality is stratified in several relatively self-contained layers which are still connected by "bottom up" and "top down" relations. Social systems are also embedded in this general layer structure. Here we may at least distinguish the following three strata:

- the *personality layer* consisting of partially genetically inherited individual constitutions and predispositions which are of potential sociological relevance but exist already *before* the entrainment into social systems like groups.

- the *micro-sociological layer* which refers to the inter individual interactions and relations which provide the necessary inner lining and background of any cooperative "synergetic" effect leading to the formation of groups. Effects making individuals ready for coöperation also consist in their conditioning by internalization of ideas like traditions and ideologies. In particular network-variables belong to this layer.
- the *macro-sociological layer* consisting of the collective macro-structures and macro-dynamics of social systems. *The formation of groups is a generic example of the emergence of such macro-patterns.* Collective personal variables and transpersonal variables like solidarity belong to this layer.

Between the micro- and macro-sociological layer there exists a *cyclical relationship, i.e. a feedback loop,* because individuals generate collective structure and dynamics (bottom–up relation), and collective structures act back on individual attitudes and decision-making (top–down relation).

Since our model belongs to the macro-sociological layer, all qualitative conceptualizations and explanations of the emergence of macro-patterns are of relevance for this model and it should be possible to relate them to the dynamics of key-variables. However, also micro-sociological theories are relevant in so far as the structure of the model (e. g. the existence of status-levels), the form of trend-functions, and the values of trend-parameters must in principle be related to the micro-sociological layer.

(b) At appropriate junctions we will have to explain some of the implications of non-linear versus linear modelling of dynamic processes since earlier quantitative approaches in sociology (as well as in economics) have used *linear models*.

(c) In discussing *different* qualitative explanatory and interpretational schemes we must take into account – as already mentioned – that they may lead to *equivalent* or at least complementary results with respect to the dynamics of a restricted set of macro-sociological key-variables like those of our model. Fararo and Doreian (1995) state that "experience with formal theories teaches that apparently very different "approaches" turn out to be complementary and, in fact, possible to coordinate and subordinate within a more comprehensive framework".

(d) Furthermore, *cyclic relationships* may appear between different explanatory schemes of group formation (e.g. between the rational choice approach, the structuralistic, and the functionalistic approach). This seems also to be the case in view of our model. However, cyclic relationships simultaneously interconnect the causation principles and relativate their absoluteness. It seems that cyclic relationships exist not only in the objective world of sociological layers but also on the "metalevel" of the theoretical understanding of causation principles.

We begin our discussion of work related to group formation with a book of Burt (1982) in which he gives a survey about *network models* as a powerful framework for social differentiation in terms of relational patterns among actors in a system. He distinguishes between *relational approaches* focussing on the relationship between pairs of actors and *positional approaches* focussing on the pattern of relations which defines the position (status) of an actor in a system of actors. That means, "all statuses are network positions in the sense of being defined by patterns of relations linking status occupants with other actors in the system".

Evidently, in particular the *positional approaches* are of potential importance for our model because they yield the micro-sociological foundation for the *status level* structure of the group configuration. Furthermore they can give justifications – or, if necessary, modifications – for the form of trend functions like payoff and obligations in state (*ih*) appearing as mathematical terms in the conditional motivation potential which in turn determines the dynamics of the status occupants. Also the relational approaches have potential importance for our model because they provide the background for direct pair interactions like persuasion activities and faith confirmation strength.

Also the work of Cook, Emerson and Gillmore (1983) belongs to the *micro-sociological* context of *network theories*. They consider theories of centrality and power distribution for small groups and validate them by computer simulations. This could also be of relevance with respect to the number and power distribution of the status levels assumed in our group configuration.

The fact criticized by Fararo and Doreian in the introductory chapter that these authors take the structural form as *given* and derive the implied distribution of power *within* the structure *without considering the dynamics* of that structure is perhaps of minor importance here because of the following reasons: The *structure* belongs to the *slowly varying macro-variables*, whereas the *power distribution* between occupants of this structure is a *micro-variable quickly adapting* to the momentary state of the structure.Because of this difference in the time scale of changes the micro-variables can be considered in each moment "to have already gone into equilibrium" with respect to the given structure.

The next author interesting for us is Homans (1950, 1958, 1974) whose work is devoted to the analysis of small groups with preferentially direct interaction between their members. Hence, his work also belongs primarily to the category of network theories although he goes beyond that. He structures the inter-individual relations by a few main *micro-sociological* variables, namely *sentiments, activities*, and *interactions*, however, already including *emergent group customs, norms*, and *practices*. He investigates the mechanisms explaining the *maintenance* of the equilibrium structure of a group as well as the elements of dynamics which he thinks to arise from a *social behavior of exchange*.

It seems, however, that in his qualitative formulations the micro-level is not clearly separated from the macro-level (which has its own quasi-autonomous dynamics). Emerging norms, customs, the ideology of a group, solidarity, etc. are still considered as directly coupled to inter-individual interactions and not as entities of their own. Nevertheless, Homans already realizes the problems of the *emergence* and not only of the stationary structure of groups.

Some of his qualitative considerations, e.g. how customs, norms, and other "transpersonal variables" (using our terminology) emerge from inter-individual interactions, could perhaps be made fruitful by translating them into equations for the evolution of transpersonal variables on the basis of the activities of individuals.

Let us now come to the influential and important work of Coleman (1964, 1990) who is one of the pioneers of mathematical sociology. Already very early he has freely used mathematical formulations. Among the mathematical formalisms used by him are stochastic equations and integrations of social choice theory with economic concepts. Thus he has demonstrated that conceptual integration instead of building barriers between sociology and economics is reasonable.

Moreover, Coleman (1990) has for instance used economic terminology in setting up a "dynamics of the linear system of action" for social exchange processes. The mathematics of his model consists in adaptation processes, namely in an *adjustment process of values* (prices) of certain goods and an *exchange process of goods* between agents. A generalization of Coleman's approach, namely a "socially embedded exchange" has recently been thoroughly discussed by Braun (1993).

Comparing Coleman's theory with our model one can find several parallels: We also make use of utility functions comprising economical as well as immaterial terms. However, his equations are *linear exchange equations* for values and goods the solution of which approach one unique equilibrium state independently of the initial state.

Our model can also be seen as a generalized exchange model – at least with respect to the collective personal variables – but our utility functions appear in a dynamic context and lead to *non-linear migratory equations* of a rather complex character.

Admittedly also a nonlinear model like ours must start from a *pre-supposed frame* of a maximum number of potentially arising groups with a pre-assumed maximum number of internal structures (e.g. status levels).

However, the non-linear dynamics is able to describe the emergence (the coming into existence) of a group from zero membership towards a finite size and also the complete decay (the disappearance) of a group. These are events of structural change!

Furthermore, it does not only depend on the chosen calibration of trend parameters but also on the chosen initial conditions which scenario will be realized. This implies that the finally approached attractor state – which could be a stable stationary state or a limit cycle or even a so-called chaotic "strange attractor" – is path-dependent and not uniquely determined independently of initial conditions as in linear theories.

We continue our consideration of related work with the discussion of authors who have, in different ways, stressed the importance of the *concept of solidarity* in sociological theory.

Let us begin with Durkheim (1964, 1973, 1915), the early thinker about the arisal and the role of solidarity in modern society. Summarizing briefly his earlier and later work, he distinguishes several ways how solidarity comes about.

In undifferentiated societies the *similarity* and *uniformity* of its members leads to a solidarity bond of common thoughts, common behavior and common culture. In differentiated and industrialized societies the *division of labor* leads to an interdependence of all members thus creating an integrative force. And even in a modern society of individualists the looming decay of integrations is halted if individualism itself is *institutionalized* to a "*cult of the individual*" which may unify the participants around common beliefs and practices.

We see that Durkheim invokes *rational necessities* (division of labor) as well as *emotional bonds* (cults, a kind of religion) to explain the emergence of solidarity. Once more it turns out that *different* rational or emotional reasons may give rise to the *same* evolution process of solidarity.

After Durkheim, research for understanding the general role of solidarity proceeds on different lines.

Parsons (1937, 1951, 1967), generalizing the ideas of Durkheim, sees solidarity as the basis of any social system and stresses the *normative character* of the sources of solidarity. He distinguishes between the two levels of common and of differentiated normative culture, where the common level is shared by all members of the society, while the differentiated level consists of *values* and *specifying norms* belonging to *functions* and *structures* of subunits of society. According to Parsons, any social structure is defined by *roles* and *collectivities, norms* and *values*.

In view of our model one could say that roles and collectivities are captured by the collective personal variables, whereas norms and values belong to the realm of transpersonal variables.

Collins (1967, 1981, 1988), also starting from Durkheim, does not follow a structuralistic–functionalistic line but in his search for the sources of solidarity he focusses on the *emotional* and *ideological* causes, e.g. when speaking of *ritual solidarity*. The elements of his "ritual model", namely co-presence, focus of attention, common emotions, membership symbols, reactions to violations and attitudes toward non-members include also (speaking in the terminology of our model) collective personal components (co-presence, reactions, attitudes) and transpersonal components (common attentions, common emotions, and membership symbols). Even more important, his concepts do already imply the *process character* of the emergence of solidarity, e.g. when speaking about the *mutual amplification* of different elements like common focus of attention and common emotional mood. In essence these arguments already amount to explaining the emergence of solidarity as a self-organization process via a feedback loop of mutually enhancing components. It seems appropriate to distinguish in the formal description of this complicated process between the *initial stage* and the *fully developed stage* of group formation.

At the initial stage of *emerging solidarity* the group is still borne and carried by a few intensely interacting individuals who *initiate solidarity*. But this emerging solidarity is still bound more or less directly to the fluctuating individual activities and interactions. However, solidarity soars up along with the growth and development of the group and there consolidates more and more its transpersonal character as an entity of its own.

In the fully developed stage of a group we have an *established solidarity* instead of an *emerging solidarity*. In this stage solidarity has become a smoothly evolving transpersonal variable representing the "groupness of the group". The *indirect coupling effects* of established solidarity can then be clearly distinguished from the *direct interactions* between the members.

Before we finish our discussion of authors' approaches to solidarity we must make a remark about a problem which lurks behind solidarity like an ever present antagonist: the *free-rider problem*, i.e. the problem why cooperation and solidarity is not destroyed or frustrated by those who abuse it. This variant of the well-known *prisoners dilemma* has been intensely treated (see for instance Axelrod, 1984; Schuessler, 1989; 1990). An advanced and more recent formal treatment of this problem relevant for group formation is that of Glance and Huberman (1993). These authors introduce the new and plausible aspect that people who cooperate have the expectation that their decision will positively influence other agents to do the same in future. However, this effect depends on the *size of the group*; it becomes effective for smaller groups only, where "outbreaks of cooperation" can occur as detailed computer simulations show. The conclusion derivable from their article is that under appropriate conditions (not too large groups) cooperation (e.g. voluntary compliance with obligations) is maintainable against abuse even without coercion.

In a macro-sociological model like ours the free-rider effect can only be taken into account in a lump manner. It is included in the saturation term of the evolution equation for solidarity. This term depends on the size of the group and reduces the attainable level of solidarity for larger groups. Therefore all trend functions which depend on the solidarity variable depend also on the solidarity-reducing influence of free-riders.

We conclude our discussion of approaches to solidarity with considerations about the book "Principles of Group Solidarity" of Hechter (1987), which is relevant for our formal modelling approach in several respects:

Hechter's approach is primarily formulated in terms of collective behavior, i.e. in terms of macro-sociological concepts. He uses concepts of rational choice theory after a concise comparative consideration of normativistic, functionalistic, and structuralistic approaches. It is of interest whether and where these concepts of causal relations in group formation can find their place in formal approaches. In his rational choice approach the question of how compliance with obligations can be attained plays a central role. We will ask how and to which degree this behavioral question can enter the macro-dynamics. Furthermore, in Hechter's approach there reappears the question of operationality of concepts and variables and the question whether or not his approach already contains a process theory, i.e. a "true dynamics" or only concepts of "comparative statics". These questions are important for any formal description of group dynamics including our model.

Hechter is satisfied neither by normativistic and functionalistic nor by structuralistic explanations of solidarity. Therefore he prefers a *rational choice approach* to group solidarity. However, then he has to cope with the problem of how solidarity and cooperation arises in a group of egoists because "rational members will seek membership in a group only if the benefit derived from access to the joint good exceeds the costs of the obligations". This problem is even sharpened because he argues against concepts of solidarity deriving from affection, fearing the lack of operationality of such concepts.

The remaining possibility of introducing a pure rationalist's operational concept of solidarity (dropping internalization of norms, emotional satisfaction, voluntary insight in group functions) is to define solidarity as a function of the extensiveness of corporate obligations and of the probability of compliance with these obligations. Here, according to Hechter, the first factor is a function of the dependence of members on the group: "the more dependent a member, the more extensive will be its obligations", and the second factor is (because of rational egoists' temptation to become a free-rider) a function of control (i.e. monitoring and sanctioning): "the greater the control the greater the compliance to obligations".

We have already mentioned how the free-rider problem is treated in our model. However, a macro-model like ours cannot reflect all psychological details, even if they are important from a micro-sociological standpoint. For instance: If *voluntary* or *coerced* compliance with obligations should lead to the *same dynamics* of the variables considered in our model, this difference could not show up anywhere in that restricted model. However, an extended model containing also micro-variables could then reveal a different fine-structure of the groups in both cases.

Hechter stresses the operationality (possibility of direct measurement) of his definition of solidarity. However, this operationality concept neglects the not easily measurable but nevertheless important solidarity creating factors and is also at the cost of the transpersonal character of solidarity. Common internalized norms, common feelings of group identity, and beliefs in group ideology establish the transpersonal character of solidarity, whereas for instance the estimations of rational egoists, in terms of how much compliance to obligations is necessary to evade sanctions, are detrimental to the formation of solidarity. For such reasons we have in our definition of the solidarity variable not insisted on its full operationality by *direct* measurements. It seems sufficient to us that the solidarity variable plays an important role in the coupled dynamic equations and thus has an *indirect* influence on the evolution of the directly measurable variables.

On the other hand it is certainly a positive aspect of Hechter's conceptualization to give the *rational choice of individuals* a central place because the decisions and actions of individual agents are the basis of group formation even if transpersonal entities eventually emerge in this process.

However, the concept of "rationality" in the decision-making of individuals should be generalized: It seems to be a fact of social psychology that norms and ideologies of a group *are* to some extent internalized by its members. This corresponds to a change of the psychological state of the individuals which implies that the estimation of benefits and costs also takes place from a new perspective. Under this perspective the realization of group purposes can become a matter of personal satisfaction, and the corresponding satisfaction terms compete – e.g. in utility functions – with terms describing personal obligations, costs, and sacrifices. *Generalized rational choice* then means that the material and immaterial benefit and cost terms of the utility of the individual member of a group depend not only on the satisfaction of immediate egoistic interests but also on the welfare of the group which is also perceived as a factor of personal satisfaction.

In our formalization of group dynamics via introduction of a conditional motivation potential we have implicitly used such a *generalized concept of rational choice*. The formal expression for this is the dependence of the motivation potential on the key-variables of the groups. That means the personal motivations depend not only on individual gains and losses in a narrow sense but on personal satisfactions or frustrations depending on the global state of the groups including their solidarity.

Finally we come back to the fundamental question which causation schemes are relevant in the formation process of groups. Our view differs from that of Hechter in so far as we do not see the decision-making individuals as the only centres of causation. Instead, we see a *cyclical relationship* between different causative factors which comprise the elements of (generalized) rational choice, structure, functions, and norms. All elements are embedded into a feedback loop which makes it impossible to isolate one element and to construct a "linear" causal nexus.

Being aware of the high complexity of cyclic causality we can only suggest a stylized cyclic causation scheme which is probably not complete:

1. Individuals coalesce in nascent groups by virtue of sharing common (material or immaterial) interests.
2. Simple collective structures are built up under individual rational aspects of construction.
3. The simple group structures bring along and carry simple functions facilitating the pursuit of the common interests.
4. Compliance with obligations is still fully voluntary and needs no norms because of the close direct interaction between the few members of the small group.
5. It is observed by the members of the growing group that the stabilizing structure can carry extended functions leading to power and influence of the group as a whole and of the individual members, but also to obligations going beyond and transforming the original interests.
6. Feelings of group identity and manifest formulations of group objectives begin to develop and rules consolidating to norms begin to be practised and entrained.
7. A transition takes place from inter-individual cooperation and rationality to transpersonal solidarity and formulations of group ideology. This facilitates further growth and efficiency of the group because direct interactions are no longer indispensable but partially substitutable by the indirect bond of group-solidarity and -ideology.
8. The loss of direct inter-individuality also favors free-riders but simultaneously stabilizes norms which, if necessary, can justify sanctions and enforce the compliance with obligations.
9. The now fully stabilizing structures, including hierarchical levels, lead to an efficient performance of functions secured by fully consolidated norms.
10. The stabilized structural, functional, and normative system of the group acts back on the members of the group by psychological internalization processes.The identification with the transpersonal superego of the group leads to a transformation of the perspectives of estimation of benefits and costs, of satisfaction and frustration in the sense that group ideals are now partially taken over as personal desires.
11. Equipped with this transformed mentality the "modified rationalists" begin a new round of organizing a new level of structure in the now fully developed group.
12. The modified mentality of the members and the reformed structures lead to the emergence of new functions and new influences of the group, etc. In total, the cyclic coupling of the causation elements of the group which simultaneously involves a "bottom up" and "top down" interaction between the micro-sociological and macro-sociological layer leads also to a slow transformation of the shape and the objective of the group.

ACKNOWLEDGEMENTS

The authors want to thank Dipl.-Phys.Richard Čalek for thoroughly revising the English text and Mrs Anja Steinhauser for her tireless care in typing the different versions of the manuscript.

REFERENCES

Axelrod, R., 1984. *The Evolution of Cooperation.* Basic Books, New York.
Braun, N., 1993. *Socially Embedded Exchange.* Verlag Peter Lang, Frankfurt.
Burt, R.S., 1982. *Toward a Structural Theory of Action.* Academic Press, New York, London.
Coleman, J.S., 1984. *Introduction to Mathematical Sociology.* MacMillan, The Free Press, New York.
Coleman, J.S., 1990. *Foundations of Social Theory.* The Belknap Press of Harvard University Press, Cambridge, MA.
Collins, R., 1981. On the micro-foundations of macrosociology. *American Journal of Sociology*, **86**, 984–1014.
Collins, R., 1985. *Three Sociological Traditions.* Oxford University Press, New York.
Collins, R., 1988. *Theoretical Sociology.* Harcourt Brace Jovanovich, San Diego.
Durkheim, E., 1915. *The Elementary Forms of Religious Life.* Free Press, New York.
Durkheim, E., 1964. *The Division of Labor in Society.* MacMillan, New York.
Durkheim, E., 1973. Individualism and the intellectuals. In R.N. Bellatz, Ed., *Emile Durkheim on Morality and Society.* University of Chicago Press, Chicago.
Emerson, R.M., K.S. Cook and M.R. Gillmore. 1983. The distribution of power in exchange networks: Theory and experimental results. *American Journal of Sociology*, **89**, 275–305.
Fararo, T.J. and P. Doreian, 1995. The theory of solidarity: an agenda of problems for mathematical sociology.

Glance, N.S. and B.A. Huberman, 1993. The outbreak of cooperation. *Journal of Mathematical Sociology*, **17**, 281–302.
Haken, H., 1977. *Synergetics – An Introduction*. Springer, Berlin, Heidelberg, New York.
Hechter, M., 1987. *Principles of Group Solidarity*. University of California Press, Berkeley.
Helbing, D., 1995. *Quantitative Sociodynamics. Stochastic Methods and Models of Social Interaction Processes*. Kluwer Academic Publishers, Dordrecht.
Homans, G.C., 1950. *The Human Group*. Harcourt Brace, New York.
Homans, G.C., 1958. Social behavior as exchange. *American Journal of Sociology*, **65**, 597–606.
Homans, G.C., 1974. *Social Behavior: Its Elementary Forms*. Harcourt Brace Jovanovich, New York.
Emerson, R.M., K.S. Cook and M.R. Gillmore. 1983. The distribution of power in exchange networks: Theory and experimental results. *American Journal of Sociology*, **89**, 275–305.
Parsons, T., 1937. *The Structure of Social Action*. Free Press, New York.
Parsons, T., 1951. *The Social System*. Free Press, New York.
Parsons, T., 1967. *Sociological Theory and Modern Society*. Free Press, New York.
Pearl, R., 1924. *Studies in Human Biology*. Williams and Wilkins, Baltimore.
Schuessler, R., 1989. Exit threats and cooperation under anonymity. *Journal of Conflict Resolution*, **33**, 728–749.
Schuessler, R., 1990. Threshold effects and the decline of cooperation. *Journal of Conflict Resolution*, **34**, 476–494.
Verhulst, P.-F., 1845. Recherches mathématiques sur la loi d'accroissement de la population. *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Bruxelles*, **18**, 1–41.
Weidlich, W., 1991. Physics and social science – the approach of synergetics. *Physics Reports*, **204**, 1–163.
Weidlich, W., 1994. Synergetic modelling concepts for sociodynamics with application to collective political opinion formation. *Journal of Mathematical Sociology*, **18**(4), 267–291.
Weidlich, W. and G. Haag (eds.). 1988. *Interregional Migration – Dynamic Theory and Comparative Analysis*. Springer, Berlin, Heidelberg, New York.
Weidlich, W. and G. Haag. 1983. *Concepts and Models of a Quantitative Sociology*. Springer, Berlin, Heidelberg, New York.