A MATHEMATICAL MODEL FOR THE BEHAVIOR OF PEDESTRIANS*

by Dirk Helbing

Institute for Theoretical Physics, University of Stuttgart, Germany

The movement of pedestrians is supposed to show certain regularities which can be best described by an "algorithm" for individual behavior and is easily simulated on computers. This behavior is assumed to be determined by an intended velocity, by several attractive and repulsive effects, and by fluctuations. The movement of pedestrians is dependent on decisions, which have the purpose of optimizing their behavior and can be explicitly modeled. Some interesting applications of the model to real situations are given, especially to formation of groups, behavior in queues, avoidance of collisions, and selection processes among behavioral alternatives.

KEY WORDS: individual pedestrians, groups, movement, dynamics, motivation, conflicts, decisions, field theory, queues, avoidance, territory, selection, break of symmetry

TYPE OF ARTICLE: mathematical model and theory.

DIMENSIONS AND UNITS: velocity of pedestrian motion per unit time; direction of pedestrian motion in two-dimensional space; and others.

1. INTRODUCTION

Human behavior is based on individual decisions. In building a mathematical model for the movement of pedestrians, one has to assume that these decisions are not completely random, but, instead, show certain regularities. This assumption may be justified, because decisions and therefore the behavior of pedestrians will usually be determined by utility maximization: a pedestrian wants to move in the most convenient way, tries to minimize delays when having to avoid obstacles and other pedestrians, intends to take an optimal path, and to walk with the minimal velocity necessary to reach a destination at a certain time, etc. The optimal behavior for a given situation can be derived by plausibility considerations and will be used as a model for pedestrian movement. Of course this optimal behavior is normally not thought about by an individual, but by trial and error he or she has automatically learned to use the most successful behavioral strategy when confronted with a standard situation (compare to sect. 3.2, (d)).

We cannot expect the model to be *exactly* valid for several reasons. First, an individual may find himself or herself in a nonstandard situation. Second, the person probably has not learned the optimal strategy yet. Third, sometimes emotional or other reasons may lead to a suboptimal behavior concerning the individual's movement. Fourth, every behavior shows a certain degree of imperfection or irregularity. All these reasons lead to deviations from optimal behavior and may be handled as fluctuations.

Nevertheless, the model gives a good impression of pedestrian movement: first, there is a tendency of pedestrians to move with an intended velocity (i.e., with an intended speed in an intended direction) (sect. 2.1). Second, individuals sometimes like to approach or avoid certain objects or persons, which can be interpreted as attractive or repulsive effects (sect. 2.2). Particularly, there is the necessity of avoiding a collision with obstacles and other pedestrians (sect. 2.2, (b)). The consequences of each aspect will be discussed in section 3 and can be compared directly with empirical observations. Some of them will be demonstrated by computer simulations (sect. 4).

*I want to thank Prof. Dr. M. R. Schroeder for giving me the chance to work in an interdisciplinary field: modeling the *social* behavior of pedestrians by *mathematical* models. Second, I am grateful to Prof. Dr. R. Kree, Prof. Dr. W. Scholl, Prof. Dr. W. Weidlich, Dr. Habil. G. Haag, and Dr. R. Reiner for their stimulating discussions. Last but not least, I am obliged to D. Weinmann, N. Empacher, K.-G. Haas, and E. Webster for reading, correcting and commenting on my manuscripts.

2. THE MODEL

2.1 Intended velocity of motion

(a) If an individual i wants to arrive at a *destination* $\vec{x}_i^0$ at time T_i, being at time t at place $\vec{x}_i(t)$, its *ideal velocity* $\vec{u}_i^0(t)$ of movement will normally have the following properties (assuming a rectilinear way to the destination as easiest situation first):

- For convenience (in order to avoid deceleration and acceleration processes), the speed should be as uniform as possible, i.e.,

$$u_i^0(t) \oplus const.$$

- In walking the remaining distance

$$s_i(t) := \| \vec{x}_i^0 - \vec{x}_i(t) \|$$

one should just use the remaining time $T_i - t$ (if one wants to avoid coming too late or too soon), i.e.,

$$u_i^0(t) := \frac{s_i(t)}{T_i - t} .$$

- The direction $\vec{e}_i$ of moving should in the simplest case be *directly* oriented toward the destination $\vec{x}_i^0$, i.e.,

$$\vec{e}_i := \frac{\vec{x}_i^0 - \vec{x}_i(t)}{\| \vec{x}_i^0 - \vec{x}_i(t) \|} .$$

All these properties are fulfilled by the ideal velocity

$$\vec{u}_i^0(t) = \frac{\vec{x}_i^0 - \vec{x}_i(t)}{T_i - t} = \frac{s_i(t)}{T_i - t} \vec{e}_i . \quad (1)$$

Intending to move with velocity $\vec{u}_i^0(t)$ guarantees a uniform movement and, when suffering deviations or delays, an orientation toward the destination and an adaptation of speed. If the available way to the destiniation is not rectilinear, it can be approximated by a polygon with edges $\vec{x}_i^n, \dots, \vec{x}_i^0$, where $\vec{x}_i^n$ denotes the starting point. In that case, the formulas above remain unaltered, but the direction $\vec{e}_i := \vec{e}_i^{\,j}$ of movement is oriented toward the next edge $\vec{x}_i^j$, after having passed the edges $\vec{x}_i^n, \dots, \vec{x}_i^{j+1}$:

$$\vec{e}_i^{\,j} := \frac{\vec{x}_i^j - \vec{x}_i(t)}{\| \vec{x}_i^j - \vec{x}_i(t) \|} .$$

Now we assume that an individual i of mass m_i, if moving with velocity $\vec{v}_i(t) := d\vec{x}_i(t)/dt$, applies a force

$$\vec{f}_i(t) \equiv m_i \frac{d\vec{v}_i(t)}{dt} := \gamma_i [\vec{v}_i^0(t) - \vec{v}_i(t)] \quad (2)$$

to get the acceleration $d\vec{v}_i(t)/dt$ toward the *intended velocity* of motion

$$\vec{v}_i^0(t) := \vec{e}_i \cdot \begin{cases} u_i^{min} & \text{for } u_i^0(t) < u_i^{min} \\ u_i^0(t) & \text{for } u_i^{min} \le u_i^0(t) \le u_i^{max} \\ u_i^{max} & \text{for } u_i^0(t) > u_i^{max} . \end{cases} \quad (3)$$

According to this assumption, the force $\vec{f}_i$ is proportional to the discrepancy $\vec{v}_i^0 - \vec{v}_i$ between intended and actual velocity, and it vanishes, when both are equal ($\vec{v}_i = \vec{v}_i^0$). By (2) $\gamma_i \vec{v}_i(t)$ approaches $\gamma_i \vec{v}_i^0(t)$ exponentially with a relaxation time of m_i/γ_i. The quantity $\gamma_i \vec{v}_i^0$ has the meaning of the *motivation to get ahead* with velocity $\vec{v}_i^0$. For $\vec{v}_i^0$ we have introduced a cutoff at u_i^{max} and u_i^{min}, because velocities above u_i^{max} are felt strenuous or uncomfortable, and velocities less than u_i^{min} are felt "boring". u_i^{min} depends on the surroundings (see (d)). In the following we will assume the common case $\vec{v}_i^0 = \vec{u}_i^0$ (i.e. $u_i^{min} \le u_i^0 \le u_i^{max}$), if nothing contrary is mentioned.

There are some other types of pedestrian movement which can be formally reduced to type (a):

(b) Suppose that individual i *plans* to pass at times t through certain places $\vec{x}_i^0(t)$. The intended velocity would then be

$$\vec{v}_i^0(t) = \frac{d\vec{x}_i^0(t)}{dt}.$$

But if the individual has, due to delays, at a certain time t_i still a distance $\Delta s_i(t_i) = \|\vec{x}_i^0(t_i) - \vec{x}_i(t_i)\|$ from the intended place $\vec{x}_i^0(t_i)$, that individual will try to make up for this distance during a time interval Δt_i, i.e. until time $t_i + \Delta t_i$. In that case, the intended velocity will, according to (1), be modified to

$$\begin{aligned}\vec{v}_i^0(t) &= \frac{\vec{x}_i^0(t_i + \Delta t_i) - \vec{x}_i(t)}{(t_i + \Delta t_i) - t} \\ &= \frac{\vec{x}_i^0(t_i + \Delta t_i) - \vec{x}_i^0(t)}{(t_i + \Delta t_i) - t} \\ &\quad + \frac{\vec{x}_i^0(t) - \vec{x}_i(t)}{(t_i + \Delta t_i) - t} \\ &\approx \frac{d\vec{x}_i^0(t)}{dt} + \frac{\vec{x}_i^0(t) - \vec{x}_i(t)}{(t_i + \Delta t_i) - t}.\end{aligned}$$

(c) If an individual i intends to move with *constant velocity* v_i^0, we get type (1) by the identification

$$u_i^{max} := v_i^0.$$

(d) Suppose individual i moves at leisure. Then he or she moves with a velocity

$$v_i^0(t) = u_i^{min}(\vec{x}_i(t)),$$

making as many interesting perceptions per time unit as intended. Therefore the appropriate velocity will depend on the actual place $\vec{x}_i(t)$. The intended direction $\vec{e}_i(t)$ of movement is given by spontaneous decisions (see section 2.2).

2.2 Contradictory motivations and decisions

An object or individual j sometimes induces a psychic reaction in a pedestrian i, motivating i to approach or avoid j (Scheflen & Ashcraft, 1976). These attractive or repulsive effects can be described by quantities $\vec{f}_{ij}^a$ or $\vec{f}_{ij}^r$ respectively, known as gradient of approach or avoidance. $\vec{f}_{ij}^{a/r}$ are not forces yet, but they are a measure for the direction and strength of the psychic motivation of i to approach or avoid j. The strength $f_{ij}^{a/r}$ of these motivations will lessen with increasing distance $r_{ij} = \|\vec{x}_j - \vec{x}_i\|$ of i and j, whereas the direction $\vec{e}_{ij}$ will be normally oriented toward or away from j, i.e.,

$$\vec{e}_{ij} = \pm\hat{r}_{ij} = \pm\frac{\vec{r}_{ij}}{r_{ij}} := \pm\frac{\vec{x}_j - \vec{x}_i}{\|\vec{x}_j - \vec{x}_i\|}$$

(+: attractive case, −: repulsive case). So with

$$\vec{r}_{ij} := \vec{x}_j - \vec{x}_i = r_{ij}\cdot\hat{r}_{ij}$$

we find

$$\vec{f}_{ij}^{a/r}(\vec{r}_{ij}) = \pm f_{ij}^{a/r}(\vec{r}_{ij})\cdot\hat{r}_{ij}. \quad (4)$$

In the absence of other motivations, the total effect

$$\vec{f}_{ij}(\vec{r}_{ij}) := \vec{f}_{ij}^a(\vec{r}_{ij}) + \vec{f}_{ij}^r(\vec{r}_{ij})$$

induced by j would play an analogous role as the motivation $\gamma_i\vec{v}_i^0$ to get ahead in equation (2) (Miller, 1944; Miller, 1959). This would lead to a movement according to

$$m_i\frac{d\vec{v}_i(t)}{dt} = \vec{f}_i(t) := \vec{f}_{ij}(\vec{r}_{ij}(t)) - \gamma_i\vec{v}_i(t). \quad (5)$$

If individual i is subject to *a couple* of motivations, the total effect would be the sum of all, resulting in the following equation of motion generalizing (2) and (5):

$$\begin{aligned}m_i\frac{d\vec{v}_i(t)}{dt} &= \vec{f}_i(t) \\ &:= \left(\sum_j \vec{f}_{ij}(t) + \gamma_i\vec{v}_i^0(t)\right) \\ &\quad - \gamma_i\vec{v}_i(t). \quad (6)\end{aligned}$$

But often it is not optimal to behave according to (6), namely in the case of contradictory motivations $\vec{f}_{ij}, \gamma_i\vec{v}_i^0$, which evoke a psychic *conflict*. Then it will be better for the individual to take a *decision*, whereby the behavioral alternative with the maximal *utility* will be preferred (Luce, 1959; Laux, 1982). In some cases this behavioral alternative can be a *compromise*. In other cases, namely when the alternatives in question mutually exclude each other, it will correspond to the alternative which provides the *strongest* motivation. We now follow Lewin's (1951) "field theoretical" view: Once a decision is taken, a *new* motivation

$$\vec{f}_i^0(t) \equiv \vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i\vec{v}_i^0(t), t\right)$$

arises as a substitute of the original motivations $\vec{f}_{ij}, \gamma_i\vec{v}_i^0$. This motivation is some kind of *psychic tension*, which causes the individual to act toward its aim in order to *diminish* this tension. In the case of pedestrians, the body will be induced to generate a *physical force*

$$\vec{f}_i(t) := \vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i\vec{v}_i^0(t), t\right) - \gamma_i\vec{v}_i(t),$$

which then causes a movement according to

$$\begin{aligned}m_i\frac{d\vec{v}_i(t)}{dt} &= \vec{f}_i(t) \\ &= \vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i\vec{v}_i^0(t), t\right) \\ &\quad - \gamma_i\vec{v}_i(t) \quad (7)\end{aligned}$$

(compare to (2), (6)). Due to (7), a pedestrian will stop moving only, when the motivation to move is vanishing, i.e., when

$$\vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i\vec{v}_i^0(t), t\right) = \vec{0}. \quad (8)$$

By

$$\vec{f}_i^0\left(\vec{f}_{ij}(t), \gamma_i\vec{v}_i^0(t), t\right) := \sum_j \vec{f}_{ij} + \gamma_i\vec{v}_i^0$$

(6) can be interpreted as special case of (7), being valid as long as no decision is taken. In that case (8) has the form of an equilibrium condition for the motivations $\vec{f}_{ij}, \gamma_i\vec{v}_i^0$:

$$\sum_j \vec{f}_{ij}(t) + \gamma_i\vec{v}_i^0(t) = \vec{0}. \quad (9)$$

Now two examples for situations will be given, in which conflicts between several motivations occur:

(a) Joining behavior

Suppose individual i perceives an attractive object or individual j of attraction $f_{ij}^a(t_{ij})$ at time t_{ij}. Individual i will then spontaneously decide to meet j, if there is enough time to do so. We assume this to be the case if

$$f_{ij}^a(t_{ij}) > \gamma_i v_i^0(t_{ij}) = \gamma_i\frac{s_i(t_{ij})}{T_i - t_{ij}},$$

i.e., if the motivation f_{ij}^a for joining j is greater than the motivation $\gamma_i v_i^0$ to continue walking (see (2)). (Here, we have made the simplification that there is only a small detour necessary to meet j.)

Individual i will stay at the meeting point for a time τ_{ij} and will leave at the moment $t_{ij} + \tau_{ij}$, when the tendency f_{ij}^a to join the attractive person or object j becomes less than the increasing tendency $\gamma_i v_i^0$ to get ahead. ($v_i^0(t)$ is growing according to the delay τ_{ij} resulting from the stay.) This condition can be written in the form

$$\begin{aligned}f_{ij}^a(t_{ij} + \tau_{ij}) &\stackrel{!}{=} \gamma_i v_i^0(t_{ij} + \tau_{ij}) \\ &= \gamma_i\frac{s_i(t_{ij})}{T_i - (t_{ij} + \tau_{ij})} \quad (10)\end{aligned}$$

(see (1)) because of $s_i(t_{ij} + \tau_{ij}) = s_i(t_{ij})$. By (10) the staying time τ_{ij} can be calculated as

$$\begin{aligned}\tau_{ij} &= (T_i - t_{ij}) - \frac{\gamma_i s_i(t_{ij})}{f_{ij}^a} \\ &= (T_i - t_{ij})\frac{f_{ij}^a - \gamma_i v_i^0(t_{ij})}{f_{ij}^a}, \quad (11)\end{aligned}$$

if f_{ij}^a is constant with time ($f_{ij}^a(t) = f_{ij}^a$).

If $f_{ij}^a(t_{ij}) \leq \gamma_i v_i^0(t_{ij})$ or, equivalently, $\tau_{ij} \leq 0$, there is not enough time to join j, and individual i will do best to continue walking without changing direction.

Summarizing (a), the decision of individual i leads to a new motivation

$$\vec{f}_i^0\left(f_{ij}^a(t), \gamma_i \vec{v}_i^0(t)\right) := \gamma_i \vec{v}_i^0(t)\Theta\left(f_{ij}^a(t) < \gamma_i v_i^0(t)\right),$$

which substitutes the contradictory motivations $\vec{f}_{ij}^a$ and $\gamma_i \vec{v}_i^0$. Here, we have introduced the decision function

$$\Theta(x) := \begin{cases} 1 & \text{if } x \text{ is true} \\ 0 & \text{if } x \text{ is false.} \end{cases}$$

Of course, individual i will change direction of motion temporarily from $\vec{e}_i(t_{ij})$ to $\vec{r}_{ij}$, if this is necessary to join j.

(b) Avoidance behavior

Suppose individual i, e.g., in order to avoid a collision, decides at time t_i to avoid an object or individual j (i.e., to keep a certain distance). Then, on the one hand, individual i tries to minimize the maximal repulsive effect against j, namely

$$\max_t f_{ij}^r(\vec{r}_{ij}(t)) =: f_{ij}^r(\vec{r}_{ij}(t_{ij})),$$

which normally occurs at the moment t_{ij} of greatest approach $r_{ij}(t_{ij})$. On the other hand, he or she wants to minimize the increase of the pressure $\gamma_i v_i^0$ to get ahead, i.e., to minimize the detour, which is necessary to avoid j. The best compromise will be to take a way, for which the maximal repulsive tendency and the tendency to get ahead have equal amounts, namely for which

$$f_{ij}^r(\vec{r}_{ij}(t_{ij})) = \gamma_i v_i^0(t_{ij}), \qquad (12)$$

and to take a rectilinear path. This path is given as tangent to the area

$$\mathcal{T}_{ij}(t) := \{\vec{x}: f_{ij}^r(\vec{x}_j(t) - \vec{x}) > \gamma_i v_i^0(t)\}, \qquad (13)$$

which describes the *territory* of j, that is *respected*, i.e., not entered by individual i. Due to (13) the area of the respected territory $\mathcal{T}_{ij}(t)$ decreases with increasing intended velocity $v_i^0(t)$ or, equivalently, with increasing pressure $\gamma_i v_i^0(t)$ to get ahead.

For the sake of completion, we assume the following additional laws of pedestrian avoidance behavior:

- When avoiding a pedestrian or obstacle j, individual i will keep his or her intended speed $v_i^0(t_i)$, changing only the intended direction from $\vec{e}_i(t_i)$ to
$$\frac{\vec{x}_i^0(t_{ij}) - \vec{x}_i(t_i)}{\|\vec{x}_i^0(t_{ij}) - \vec{x}_i(t_i)\|},$$
where $\vec{x}_i^0(t_{ij})$ is the intended position of i for the moment of greatest approach. According to this, the motivation to get ahead will be changed from $\gamma_i \vec{v}_i^0(t_i)$ to
$$\vec{f}_i^0\left(\vec{f}_{ij}^r(t), \gamma_i \vec{v}_i^0(t)\right) := \gamma_i v_i^0(t)\frac{\vec{x}_i^0(t_{ij}) - \vec{x}_i(t_i)}{\|\vec{x}_i^0(t_{ij}) - \vec{x}_i(t_i)\|}$$
during the time it takes to avoid j (i.e. for times t with $t_i \leq t \leq t_{ij}$).
- An individual i reacts a time $\Delta t_{ij} := t_{ij} - t_i$ before a collision would be expected. This time Δt_{ij} is a psychic parameter, which, of course, will be the greater the larger the dimension of the obstacle j is. The *distance* d_{ij} of reaction before the location of a probable collision is the product of Δt_{ij} and speed v_i:
$$d_{ij} = v_i \Delta t_{ij}.$$

It is plausible that the necessary angular change of direction when avoiding an obstacle j will be the greater, the lower the distance d_{ij} of the obstacle j is. So the (average) change of direction will be the greater, the lower the (average)

speed is. This can be observed when comparing more and less crowded situations.

- If the distance for passing j on the left is nearly the same as for passing j on the right, we assume individual i will take the right hand side with probability p_1 and the left hand side with probability $p_2 := 1 - p_1$.
- But if there is no chance of passing j, e.g., when the way is too crowded, individual i will decelerate (as long as necessary) to a velocity $\vec{v}_i$, which allows a maximal component $\vec{v}_i \cdot \vec{e}_i$ of movement into the intended direction $\vec{e}_i$. This maximal component is normally equal to the component $\vec{v}_j \cdot \vec{e}_i$, which the hindering pedestrian's velocity $\vec{v}_j$ has in direction $\vec{e}_i$ (corresponding to the situation, that individual i walks in a gap behind a pedestrian j with velocity $\vec{v}_j$). However, if pedestrian j has an opposite direction with respect to i ($\vec{v}_j \cdot \vec{e}_i < 0$), it will be better for individual i to stop ($\vec{v}_i = \vec{0}$). Summarizing these results we have the relation
$$\vec{v}_i \cdot \vec{e}_i := \begin{cases} \vec{v}_j \cdot \vec{e}_i & \text{if } \vec{v}_j \cdot \vec{e}_i > 0 \\ 0 & \text{else.} \end{cases}$$

3. CONCLUSIONS AND COMPARISON WITH REAL SITUATIONS

3.1 Effects of the intended velocity of motion

(a) Velocity of motion

According to (2) a pedestrian would normally walk with velocity $\vec{v}_i(t) \approx \vec{v}_i^0(\vec{x}_i(t))$. But in order to avoid collisions, an individual i suffers detours or delays, and as a consequence, its *smoothed* velocity $\vec{\bar{v}}_i(t)$ of motion will probably have the more general form

$$\frac{d\vec{\bar{x}}_i(t)}{dt} = \vec{\bar{v}}_i(t) \approx \vec{w}_i + k_i \vec{v}_i^0\left(\vec{\bar{x}}_i(t)\right) = \vec{w}_i + k_i \frac{\vec{x}_i^0 - \vec{\bar{x}}_i(t)}{T_i - t} \qquad (14)$$

with $k_i \leq 1$ (see (1)). k_i and $\vec{w}_i$ are empiric parameters depending on the walking situation and describing the effect of "interindividual interactions". (14) can be solved by

$$\vec{\bar{v}}_i(t) = \vec{w}_i + \begin{cases} \left(\dfrac{k_i}{T_i - t_i^0}\left[\vec{x}_i^0 - \vec{\bar{x}}_i(t_i^0)\right] + \dfrac{k_i}{k_i - 1}\vec{w}_i\right) \cdot \left(1 - \dfrac{t - t_i^0}{T_i - t_i^0}\right)^{k_i - 1} - \dfrac{k_i}{k_i - 1}\vec{w}_i & \text{if } k_i \neq 1 \\ \dfrac{1}{T_i - t_i^0}\left[\vec{x}_i^0 - \vec{\bar{x}}_i(t_i^0)\right] + \ln\left(1 - \dfrac{t - t_i^0}{T_i - t_i^0}\right) \cdot \vec{w}_i & \text{if } k_i = 1, \end{cases}$$

where t_i^0 is the time when individual i starts walking. We can conclude the following:

- If the smoothed actual velocity $\bar{v}_i$ is less than the intended velocity v_i^0, then $\bar{v}_i$ and v_i^0 will be growing with time, because from (14)
$$\frac{d\vec{\bar{v}}_i(t)}{dt} \approx k_i \frac{\vec{v}_i^0(t) - \vec{\bar{v}}_i(t)}{T_i - t} = k_i \frac{d\vec{v}_i^0(t)}{dt}$$
can be derived. So individual i will speed up in the course of time unless the maximal velocity $\vec{v}\max_i = \vec{w}_i + k_i u_i^{max} \vec{e}_i$ is reached. (Apart from (14) we have now taken into account eq. (3).)
- Individual i will arrive at destination $\vec{x}_i^0$ too late if the smoothed actual velocity $\bar{v}_i(t)$ would have to exceed the maximal velocity v_i^{max} before time T_i, i.e., if
$$\lim_{t \to T_i} \bar{v}_i(t) > v_i^{max}.$$

- Individual i will keep less distance from other pedestrians j as $\bar{v}_i(t)$ increases (see sect. 2.2, (b)), because

$$\gamma_i \bar{v}_i^0(t) = \gamma_i \frac{\vec{\bar{v}}_i(t) - \vec{w}_i}{k_i}$$

and equation (12). Individual i then shows less respect for the "territory" of individual j: he or she walks more aggressively and perhaps even pushes.

- In crowded situations individual i can prevent having to hurry by intending to walk with velocity

$$\vec{v}_i^0 := -\frac{\vec{w}_i}{k_i} + \frac{1}{k_i}\frac{\vec{x}_i^0 - \vec{x}_i(t)}{T_i - t}.$$

This strategy will lead to a smoothed actual velocity of $\vec{\bar{v}}_i = \vec{v}_i^0$.

(b) Effect of an unexpected detour

In some situations individual i has to take an unexpected detour Δs_i, e.g., if he or she has forgotten something and has suddenly remembered this at time t_i^+. So the intended velocity changes according to (1) from

$$v_i^0(t_i^-) = \frac{s_i(t_i^-)}{T_i - t_i^-}$$

at the preceding moment t_i^- to

$$v_i^0(t_i^+) = \frac{s_i(t_i^+)}{T_i - t_i^+} = \frac{s_i(t_i^-) + \Delta s_i}{T_i - t_i^+} > v_i^0(t_i^-).$$

By (2) this gives rise to a sudden increase of velocity v_i, which can often be observed, especially for individuals who walk according to a plan $\vec{x}_i^0(t)$ (see sect. 2.1, (b)). These individuals try to speed up to maximal velocity $v_i^0(t_i^+) := u_i^{max}$ until they have, after a time interval

$$\Delta t_i \geq \frac{\Delta s_i}{u_i^{max} - v_i^0(t_i^-)},$$

reached their plan $\vec{x}_i^0(t_i + \Delta t_i)$ again (in the sense of $\vec{x}_i(t_i + \Delta t_i) = \vec{x}_i^0(t_i + \Delta t_i)$).

(c) Behavior in a queue

If the front of a queue has come to rest, the following phenomenon can often be observed: after a while, one of the waiting individuals begins to move forward a little, causing the successors to do the same. This process propagates in wave-like manner to the end of the queue, and the distance to move forward increases.

Why do individuals behave in such a paradoxical way? They don't move up any faster but only cause the queue to become more crowded! Our model gives the following interpretation:

At time t_i an individual i keeps a distance $r_{i,i-1}(t_i)$ to the individual $i-1$ in front, which is (according to (9) and (12)) given by

$$f_{i,i-1}^r(r_{i,i-1}(t_i)) = \gamma_i v_i^0(t_i).$$

$f_{i,i-1}^r$ is the repulsive effect describing the territory of individual $i-1$ respected by i. As we know from (1), $v_i^0(t)$ grows as time t passes, because individual i is at rest ($\vec{x}_i(t) = \vec{x}_i(t_i)$). So at time $t_i + \Delta t_i$ individual i would prefer to have a distance

$$r_{i,i-1}(t_i + \Delta t_i) =: r_{i,i-1}(t_i) - \Delta r_i(t_i + \Delta t_i),$$

which has reduced by an amount Δr_i and is given by

$$f_{i,i-1}^r(r_{i,i-1}(t_i + \Delta t_i)) = \gamma_i v_i^0(t_i + \Delta t_i). \qquad (15)$$

But individual i moves up a distance Δr_i only if

$$\Delta r_i \geq \Delta r_i^{min}, \qquad (16)$$

i.e., if the increment Δr_i exceeds a minimal stride Δr_i^{min}. So the first individual moving up is the individual i, for which condition

$$\Delta r_i(t_i + \Delta t_i) = \Delta r_i^{min}$$

is fulfilled first. This is the case at a time $t := t_i + \Delta t_i$, i.e., a time interval Δt_i after its last step at time t_i. Now the successors $i + n$ ($n \geq 1$) will move forward a distance

$$s_{i+n} = \sum_{j=i}^{i+n} \Delta r_j(t) = \sum_{j=i}^{i+n} \Delta r_j(t_j + \Delta t_j)$$

according to (15) and (16), because $s_{i+n} \geq \Delta r_{i+n}^{min}$ will normally be fulfilled.

3.2 Attractive and repulsive effects

(a) Constant density

Suppose a number of N individuals having only a negligible intention to move ($\vec{v}_i^0 \approx 0$) stay in an area of a (dining) hall, a waiting room, a beach, an underground station, etc. with size A. One can then observe a quite uniform distribution of individuals (with constant density N/A) if there are no special attractions in area A and no acquaintances among the individuals (see (b)). This is due to the repulsive effects $\vec{f}_{ij}^r$ between each pair of individuals i and j, which are in equilibrium (see (9)), when all individuals occupy a personal territory of nearly equal size.

(b) Formation of groups

If there *are* acquaintances between the individuals of example (a), a truncated Poisson distribution

$$p_k = \mathcal{N}\frac{\lambda^k}{k!} \qquad k = 1,2,\ldots \qquad (17)$$

can be found for the proportion p_k of groups consisting of k members. This distribution is well confirmed by empirical data (Coleman, 1964) and can be explained by the following mathematical model of Coleman (Coleman & James, 1961; Coleman, 1964):

$$\begin{aligned}\frac{dp_k}{dt} &= [\text{transitions from } l\ (\neq k) \text{ to } k \\ &\quad - \text{transitions from } k \text{ to } l\ (\neq k)]/\text{time unit} \\ &= \sum_{l(\neq k)} p_l \cdot r(l \to k) - \sum_{l(\neq k)} p_k \cdot r(k \to l) \\ &= (p_{k+1} \cdot (k+1) \cdot \beta + p_{k-1} \cdot \alpha \cdot p_1) \\ &\quad - (p_k \cdot k \cdot \beta + p_k \cdot \alpha \cdot p_1) \qquad (18)\end{aligned}$$

for $k = 2, 3, \ldots$, and

$$\sum_{k=1}^{\infty} p_k = 1. \qquad (19)$$

In (18) we have used

$$r(k \to l) = \begin{cases} k \cdot \beta & \text{if } l = k - 1 \\ \alpha \cdot p_1 & \text{if } l = k + 1 \\ 0 & \text{else} \end{cases}$$

with $l \geq 2$. This means that a group with k individuals loses individuals independently with rate β and gains single individuals with rate $\alpha \cdot p_1$ (which is proportional to the number of single individuals). Other transitions are assumed to be relatively unimportant.

(18), (19) have the stationary solution

$$p_k = \frac{1}{e^\lambda - 1}\frac{\lambda^k}{k!},$$

given by $dp_k/dt = 0$, where

$$\lambda := \ln\left(\frac{\alpha}{\beta} + 1\right). \qquad (20)$$

We now connect these results with our model: For β we could simply take the mean value of the reciprocal $1/\tau_{ij}$ of the time τ_{ij} which an individual i stays in a group j, because this is the rate of leaving a group (see (11)):

$$\beta := E\left(\frac{1}{\tau_{ij}}\right). \qquad (21)$$

On the other hand, α can be assumed of the form

$$\alpha := p_+ J, \qquad (22)$$

where J is the rate of recognized groups per time unit and p_+ is the probability of joining a recognized group j. According to sect. 2.2, (a), p_+ is the probability $P(\tau_{ij} > 0)$, that the staying time τ_{ij} is positive:

$$p_+ := P(\tau_{ij} > 0) = P(f_{ij}^a > \gamma_i v_i^0). \quad (23)$$

f_{ij}^a is, of course, the attractive effect between individual i and group j. Due to (20) to (23) the following conclusions can now be made:

- Parameter λ, which is a measure for the average number of members of a group, increases with the mean value of the staying time τ_{ij}, i.e., it decreases with growing intended velocity v_i^0 and increases with growing remaining time $T_i - t_{ij}$ (see (11)). This is consistent with the data (Coleman, 1964).
- If the motivation f_{ij}^a to join a group j is less than the motivation $\gamma_i v_i^0$ to get ahead for all individuals i and groups j, we have $p_+ = 0$ and $\alpha = 0$. In that case no groups are forming at all and (a) can be applied again (if $\bar{v}_i^0 \approx 0$).

(c) Superposition of attractive and repulsive effects

Often a person or object j has an attractive effect f_{ij}^a and a repulsive effect f_{ij}^r as well. As a consequence of equation (5), individual i will then show one of several characteristic dynamic behaviors known from approach-avoidance conflicts, depending on the special form of the motivation gradient $\vec{f}_{ij}(\vec{r}_{ij})$ (Herkner, 1975). Especially, for negligible intention to move ($\bar{v}_i^0 \approx 0$), individual i will prefer a certain distance (Miller, 1944; Dewdney, 1987), for which the equilibrium condition

$$f_{ij}(\vec{r}_{ij}) = f_{ij}^a(\vec{r}_{ij}) - f_{ij}^r(\vec{r}_{ij}) = 0$$

is fulfilled (see (9) and (4)), i.e., for which the attractive and the repulsive effects have equal strengths.

(d) Break of symmetry for avoidance behavior

Suppose two individuals walk in opposite directions and try to avoid each other in order not to collide. Then each tries to pass the other with probability p_1 on the right and probability $p_2 = 1 - p_1$ on the left (see sect. 2.2, (b)). The probability for avoiding each other successfully is then

$$p_1 \cdot p_1 + p_2 \cdot p_2 =: 1 - w.$$

Otherwise, with probability

$$w = p_1 \cdot p_2 + p_2 \cdot p_1 = 2 p_1 \cdot p_2 \le \tfrac{1}{2}, \qquad (24)$$

they have to try again, etc., until they pass on *different* sides. This phenomenon is well known.

The mean value $E(n)$ for the necessary number n of attempts to avoid each other is given by

$$E(n) = \sum_{n=1}^{\infty} n \cdot w^{n-1} \cdot (1 - w) = \frac{1}{1 - w}. \qquad (25)$$

Taking (24) into account, this expression is *maximal* for $w = 1/2$, i.e., for symmetric probabilities

$$p_1 = p_2 = \tfrac{1}{2}$$

of avoidance for both sides. (25) is *minimal* for $p_1 = 0$ or $p_1 = 1$ (deterministic behavior!). Therefore *asymmetric* probabilities $p_1 \neq p_2$ of avoidance are favorable. In fact, in most countries individuals more frequently pass other individuals on the right ($p_1 > 1/2$). As a consequence, crowded ways often show two different lanes of opposite direction, which stick to the right side respectively

(Navin & Wheeler, 1969; Oeding, 1963; Older, 1968). This behavior reduces the frequency of situations of avoidance and corresponding delays.

Selection of one behavioral alternative

For an explanation of the break of symmetry ($p_1 \neq p_2$), we consider the following general model which describes the temporal change of the proportion p_k of individuals showing a certain behavioral alternative k (compare to Eigen & Schuster, 1979):

$$\frac{dp_k}{dt} = \sum_l [M_{kl} - s_k \delta_{kl}] p_l + \xi_k. \quad (26)$$

M_{kl} are *mutation* rates from behavior l to behavior k per time unit and person. s_k will be chosen as

$$s_k = \sum_m M_{mm} p_m + \sum_{m(\neq k)} M_{mk} \quad (27)$$

to guarantee $\sum_k dp_k/dt = 0$, which is necessary for normalization ($\sum_k p_k = 1$). $-s_k \delta_{kl} p_l$ has the effect of a *selection* between the behavioral alternatives k. ξ_k are random fluctuations of the proportion p_k of behavioral alternative k. For the problem of avoidance we have only two alternatives: one to pass a hindering pedestrian on the right ($k := 1$), and the other to pass on the left ($k := 2$). As mutation matrix we take

$$\underline{M} := \underline{A} + \underline{B} \qquad (28)$$

with

$$\underline{A} := \lambda \begin{pmatrix} p_1 & 1 - p_2 \\ 1 - p_1 & p_2 \end{pmatrix} \qquad (29)$$

and

$$\underline{B} := \beta \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}. \qquad (30)$$

According to $\underline{A}$, a behavioral alternative k becomes more probable (by learning), the greater the proportion p_k of individuals with behavior k is (because in our case behavior k is the more *successful* the more often it occurs (Skinner 1938, 1953). On the other hand, $\underline{B}$ describes a random choice of some behavior k with probability $1/2$ due to trial (and error). (The individual behavior depends on the respective situation.) Substitution of (27) to (30) in (26) now gives

$$\frac{dp_k}{dt} = [2\lambda p_k \cdot (1 - p_k) - \beta] \cdot \left(p_k - \frac{1}{2}\right) + \xi_k, \qquad (31)$$

which, for $\beta \ge \lambda/2$, has the only stationary solution $p_k = 1/2$. However, for a low tendency β to choose the behavior randomly ($0 \le \beta < \lambda/2$), (31) has three stationary solutions: $p_k = 1/2$, being *unstable* against fluctuations ξ_k, and $p_k = 1/2 \cdot (1 \pm \sqrt{1 - 2\beta/\lambda})$, being *stable*! As a consequence of the instability of $p_k = 1/2$, fluctuations will cause the proportion p_k to tend either toward $p_k = 1/2 + 1/2\sqrt{1 - 2\beta/\lambda}$ (preferring the right side) or toward $p_k = 1/2 - 1/2\sqrt{1 - 2\beta/\lambda}$ (preferring the left side). By spatial diffusion of this learning process the preferred behavior is spread over wide areas (e.g., countries) and stabilized against crossing $p_k = 1/2$, which could in principal be induced by fluctuations.

We now assume that individual i *overtakes* pedestrian j walking in the *same* direction. Here, we normally do not have to expect any complications from the behavior of j. So the avoidance behavior will be successful with probability $w = 1$, regardless of the side of passing. Our mutation matrix $\underline{M}$ then will not depend on the proportions p_1, p_2 of pedestrians passing on the left or on the right ($\lambda = 0$). This time we have the equation

$$\frac{dp_k}{dt} = -\beta\left(p_k - \frac{1}{2}\right) + \xi_k$$

(see (31)), which has only one stationary solution: the symmetric probability $p_k = 1/2$ of avoidance, which is stable.

4. COMPUTER SIMULATIONS

In order to test the somewhat algorithmical model of section 2 (especially section 2.2, (b)), some simple computer simulations have been carried out. The corresponding computer program works as follows:

- First, the geometrical configuration is determined (e.g., a normal pedestrian way or a pedestrian way with several obstacles).
- In the examples presented, two types (i.e., main directions) of motion are necessary: pedestrians intending to walk from the left to the right are represented by black lines, those intending to walk in the opposite direction are represented by gray lines. Every line has the meaning of an individual's actual stride, and its length is proportional to its velocity.
- As initial configuration, a statistically uniform spatial distribution of N pedestrians is taken ($N = 350$ or 500), one-half belonging to the black type of motion, the other half belonging to the gray type (see Fig. 1). The intended speeds of each direction are distributed by chance (Gaussian), whereby the same mean speeds and the same velocity variances were chosen for both directions of motion.
- At the beginning of the simulation, a certain order of the N pedestrians is chosen at random. The pedestrians take each step according to that order. After even the Nth pedestrian has taken his or her Sth step, the first pedestrian is taking his or her $(S+1)$st one. For each individual leaving on one side of a figure, an equivalent one enters on the other side, i.e., the right side of each figure can be assumed to be connected to the left side (periodic boundary conditions).

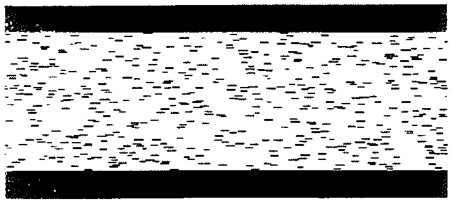

FIG. 1. ($N = 500$, $S = 0$): Initial configuration: N pedestrians with varying speeds are distributed randomly over a pedestrian way, the black ones walking from left to right, the grey ones walking in opposite direction.

- Now the considerations from section 2.2, (b) are taken into account: a pedestrian taking the next step will move by his or her intended stride into the intended direction, if this is possible. If not, i.e., if the individual would have to cross another pedestrian's step, he or she will change direction by an angle, which will be the greater, the nearer the hindering pedestrian is. However, if even this does not prevent one pedestrian from crossing another pedestrian's step, the intended stride will be made as short as necessary, possibly leading to a stop. In the case of a change of direction, the right side is chosen with probability

$$p_1 := \begin{cases} 1/2 & \text{if both pedestrians belong to the same direction of motion} \\ p & \text{if the pedestrians belong to different directions of motion.} \end{cases}$$

The left side is chosen with probability $p_2 = 1 - p_1$.
- If a pedestrian comes into the proximity of an obstacle, he or she temporarily changes the intended direction, preferring to pass the obstacle at the nearest side in order to suffer the least possible detour. If both sides have approximately the same distance, each side is chosen with probability 1/2.

The computer simulations show the following results:

- For symmetric avoidance behavior ($p = 1/2$), changes of direction appear very often, because encounters of pedestrians from opposite directions are likely to occur everywhere (see Fig. 2). In the case of *asymmetric* avoidance behavior ($p = 0.7$), two walking lanes of opposite direction develop in the course of time (see Fig. 3). Obviously, there are less changes of direction necessary, occurring mainly at the borderline between the opposite lanes.
- In the presence of an obstacle, a pedestrian free area develops in front of and behind the obstacle (see Figures 4 and 5). But, whereas an obstacle in the middle of a pedestrian way causes only a small area not to be used (see Fig. 4), obstacles at the margin do reduce the effective width over a long distance (see Fig. 5).

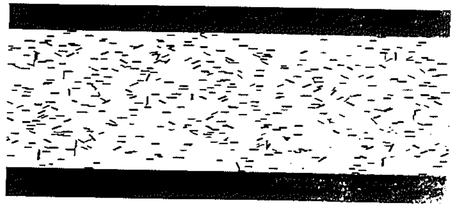

FIG. 2. ($N = 500$, $S = 500$, $p = 1/2$): In order to avoid collisions with other pedestrians the direction of walking has to be changed often.

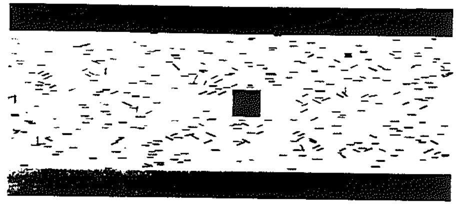

FIG. 4. ($N = 350$, $S = 540$, $p = 0.7$): In front of and behind an obstacle a pedestrian free area develops.

5. CONCLUSIONS

We have set up a model for the movement of pedestrians starting from the idea that individual decisions are guided by maximization of utility. Once a decision is taken, a special kind of psychic motivation or tension to realize this decision arises, which causes the individual to act in order to neutralize the psychic tension. For example, when individual i wants to reach a certain destination at time T_i, it would be best to walk with a suitable velocity $\vec{v}_i^0$. So the pedestrian will decide to walk with the "intended velocity" $\vec{v}_i^0$, applying a physical force $\vec{f}_i$, which vanishes when the pedestrian's actual velocity $\vec{v}_i$ is equal to the intended one. In the case of delays, the intended velocity has to be corrected upward in the course of time, causing the pedestrian to speed up and perhaps to walk more aggressively. Waiting in a queue that has come to rest, an individual will instead move forward after some time, which motivates the successors to move forward, too. Therefore, this behavior propagates in a wave-like manner to the end of the queue and leads to a more crowded queue.

In addition, a pedestrian is subject to attractive or repulsive influences, approaching or avoiding certain individuals or things j. If, for example, the motivation $\vec{f}_{ij}^a$ to approach some person (say a friend) or some object (e.g., a shop window) is greater than the motivation to get ahead, the pedestrian i will decide to join this individual or object for a while, but will leave the moment at which the motivation to join the attractive person

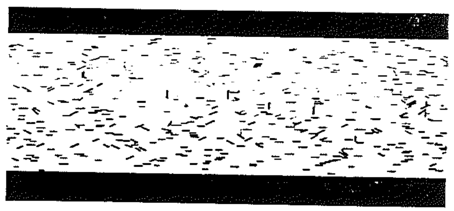

FIG. 3. ($N = 500$, $S = 500$, $p = 0.7$): If the probability p for passing a hindering pedestrian on the right is different from the probability $1 - p$ for passing it on the left, two lanes of opposite direction develop.

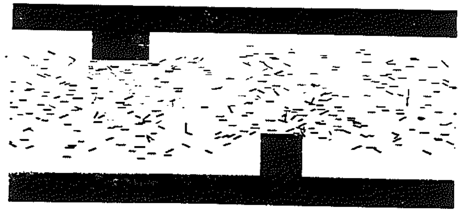

FIG. 5. ($N = 350$, $S = 540$, $p = 0.7$): Obstacles at the margin of a pedestrian way reduce its effective width.

or object j becomes less than the increasing motivation to get ahead with the intended velocity (which is growing according to the delay resulting from the stay). If, right from the beginning, the motivation of a pedestrian to get ahead is greater than the motivation to join a certain person or object j, the pedestrian's best decision will be not to change path at all. This model leads to a detailed description of group formation.

However, there are also repulsive effects f_{ij}^r. They describe, for example, the personal territories of individuals j. As a consequence, individuals who don't know each other normally spread uniformly in an area of a hall, a waiting room, a cafe, a beach, etc. (if there are no special attractions). In situations where pedestrian i has to avoid another one j in order to prevent a collision, the pedestrian prefers to suffer only a minimal detour. So individual i will pass individual j along a tangent to the territory of j respected by i. This respected territory is given as the area around j, for which the repulsive effect f_{ij}^r of j is greater than the motivation $\gamma_i v_i^0$ of i to get ahead with speed v_i^0.

Mathematically, it appears to be favorable when most pedestrians prefer either the right side or the left side when passing each other. This results in the development of walking lanes in pedestrian crowds. With both sides being equivalent, one side will be used by a growing majority, once it has been chosen at random. This is one example being representative for many others, where the most successful or most efficient behavior is adopted by trial and error, causing a selection between behavioral alternatives.

After having set up a "microscopic" model, i.e., one for the movement of *individuals*, one may be interested in a model for a great number of interacting pedestrians. Such a model is developed in Helbing (1991). It shows some similarities to gaskinetic and fluid dynamic equations, but contains some additional terms that are characteristic for pedestrian movement

REFERENCES

Coleman, J. S. *Introduction to mathematical sociology*. New York: The Free Press of Glencoe, 1964, 361-375.

Coleman, J. S. & James, J. The equilibrium size distribution of freely forming groups. *Sociometry*, 1961, 24, 36-45.

Dewdney, A. K. Diverse personalities search for social equilibrium at a computer party. *Scientific American*, 9/1987, 257, 104-107.

Eigen, M. & Schuster, P. *The hypercycle*. Berlin et al.: Springer, 1979, 8-10.

Helbing, D., A fluid dynamic model for the movement of pedestrians. Submitted to *Physics of Fluids A*, 1991.

Herkner, W. H. Ein erweitertes Modell des Appetenz-Aversions-Konflikts (A generalized model for the approach-avoidance conflict). *Z. Klin. Psychol.*, 1975, 4, 50-60.

Laux, H. *Entscheidungstheorie* (Decision theory). Berlin et al.: Springer, 1982.

Lewin, K. *Field theory in social science*. New York: Harper & Cruthers, 1951.

Luce, R. D. *Individual choice behavior*: A theoretical analysis. New York: Wiley, 1959.

Miller, N. E. Experimental studies of conflict. In: Hunt, J. McV (ed.) *Psychology: Personality and the behavioral disorders*, Vol. 1. New York: Ronald, 1944.

Miller, N. E. Liberalization of basic S-R concepts: Extension to conflict behavior, motivation, and social learning. In: Koch, S. (ed.) *Psychology: A study of science*, Vol. 2. New York: McGraw Hill, 1959.

Navin, P. D. & Wheeler, R. J. Pedestrian flow characteristics. *Traffic Engineering*, 1969, 39, 31-36.

Oeding, D. *Verkehrsbelastung und Dimensionierung von Gehwegen und anderen Analagen des Fußgängerverkehrs (Traffic volume and sizing of footpaths and other constructions of pedestrian traffic)*. Bonn: Straßenbau and Straßenverkehrstechnik 22, 1963, 30, 8: Fig. 10.

Older, S. J. Movement of pedestrians on footways in shopping streets. *Traffic Engineering and Control*, 1968, 10, 160-163.

Scheflen, A. E. & Ashcraft, N. *Human territories: How we behave in space-time*. Englewood Cliffs: Prentice-Hall, 1976.

Skinner, B. F. *The behavior of organisms*. NewYork: Appleton, 1938.

Skinner, B.F. *Science and human behavior*, New York: Macmillan, 1953.

(Manuscript received January 25, 1991)