G. Haag U. Mueller K. G. Troitzsch (Eds.)

Economic Evolution and Demographic Change

Formal Models in Social Sciences

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

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ISBN 3-540-56172-2 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-56172-2 Springer-Verlag New York Berlin Heidelberg

Printed in the U.S.A.

Typesetting: Camera ready by author/editor
42/3140-543210 - Printed on acid-free paper

18 A Mathematical Model for Behavioral Changes by Pair Interactions

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18.1 Introduction

This paper treats a mathematical model for the change of the fraction $P(i,t)$ of individuals who show a certain behavior i. Models of this kind are of great interest for a *quantitative understanding* or *prognosis* of social developments. For the description of the competition or cooperation of populations there already exist *game theoretical approaches* (see, for example, Mueller (1990), Axelrod (1984), von Neumann and Morgenstern (1944), Luce and Raiffa (1957)). However, the model devoloped in this paper shows to be more general, since it includes as special cases

- not only the *game dynamical equations* (Hofbauer and Sigmund (1988)), but also
- the *logistic equation* (Verhulst (1845), Pearl (1924), Helbing (1992)),
- the *Gravity model* (Ravenstein (1876), Zipf (1946)),
- the LOTKA-VOLTERRA equations (Lotka (1920, 1956), Volterra (1931), Hofbauer (1981), Goel et. al. (1971), Hallam (1986), Goodwin (1967)), and
- the quantitative social models of Weidlich and Haag (Weidlich & Haag (1983, 1988), Weidlich (1991)).

This model assumes behavioral changes to occur with a certain *probability* per time unit, called the *transition rate*. The transition rate is decomposed into

- a rate describing *spontaneous* behavioral changes, and
- a rate describing behavioral changes due to *pair interactions* of individuals.

Three different kinds of pair interactions can be distinguished:

- First, *imitative processes*, which describe the tendency to take over the behavior of another individual.
- Second, *avoidance processes*, causing an individual to change the behavior if meeting another individual with the same behavior.
- Third, *compromising processes*, which describe the readiness to change the behavior to a new one when meeting an individual with another behavior.

Representative solutions of the model are illustrated by computer simulations. By distinguishing several *subpopulations* a, different *types* of behavior can be taken into account.

As one would expect, there is a connection of this model with the game dynamical equations. In order to establish this connection, the transition rates have to be taken in a special way which depends on the *expected success* of the behavioral strategies. The essential effect is given by imitative processes.

A stochastic version of game theory is formulated, from which the ordinary game dynamical equations follow as equations for the *most probable* behavioral distribution. An example of two equivalent competing strategies serves as an illustration of these equations and allows the description of the *selforganization* of a behavioral convention.

18.2 The master equation

Suppose, we have a social system with N individuals. These individuals can be divided into A *subpopulations* a consisting of N_a individuals, i.e.,

$$\sum_{a=1}^{A} N_a = N .$$

By subpopulations different social groups (e.g. blue and white collars) or different characteristic *types* of behavior are distinguished.

The N_a individuals of each subpopulation a are distributed over several *states*

$$i \in \{1, \dots, S\} ,$$

which represent the *behavior* or the (behavioral) *strategy* of an individual. If the *occupation number* n_i^a denotes the number of individuals of subpopulation a who show the behavior i, we have the relation

$$\sum_{i=1}^{S} n_i^a = N_a . \tag{18.1}$$

Let

$$\vec{n} := (n_1^1, \dots, n_S^1, \dots, n_i^a, \dots, n_1^A, \dots, n_S^A)$$

be the vector consisting of all occupation numbers n_i^a. This vector is called the *socioconfiguration*, since it contains all information about the distribution of the N individuals over the states i. $P(\vec{n},t)$ shall denote the *probability* to find the socioconfiguration $\vec{n}$ at time t. This implies

$$0 \le P(\vec{n},t) \le 1 \quad \text{and} \quad \sum_{\boldsymbol{n}} P(\vec{n},t) = 1 .$$

If transitions from socioconfiguration $\vec{n}$ to $\vec{n}'$ occur with a probability of $P(\vec{n}',t+\Delta t|\vec{n},t)$ during a short time interval Δt, we have a *(relative) transition rate* of

$$w(\vec{n}',\vec{n};t) := \lim_{\Delta t \to 0} \frac{P(\vec{n}',t+\Delta t|\vec{n},t)}{\Delta t} .$$

The *absolute* transition rate of changes from $\vec{n}$ to $\vec{n}'$ is the product $w(\vec{n}',\vec{n};t)P(\vec{n},t)$ of the probability $P(\vec{n},t)$ to have the configuration $\vec{n}$ and the *relative* transition rate $w(\vec{n}',\vec{n};t)$ if having the configuration $\vec{n}$. Whereas the *inflow* into $\vec{n}$ is given as the sum over all absolute transition rates of changes from an *arbitrary* configuration $\vec{n}'$ to $\vec{n}$, the *outflow* from $\vec{n}$ is given as the sum over all absolute transition rates of changes from $\vec{n}$ to *another* configuration $\vec{n}'$. Since the temporal change of the probability $P(\vec{n},t)$ is determined by the inflow into $\vec{n}$ reduced by the outflow from $\vec{n}$, we find the *master equation*

$$\begin{aligned} \tfrac{d}{dt}P(\vec{n},t) &= \text{inflow into } \vec{n} &&- \text{outflow from } \vec{n} \\ &= \textstyle\sum_{\boldsymbol{n}'} w(\vec{n},\vec{n}';t)P(\vec{n}',t) &&- \textstyle\sum_{\boldsymbol{n}'} w(\vec{n}',\vec{n};t)P(\vec{n},t) \end{aligned} \tag{18.2}$$

(see Haken (1983)), which is a *stochastic equation.*

It shall be assumed that two processes contribute to a change of the socioconfiguration $\vec{n}$:

- Individuals may change their behavior i spontaneously and independently of each other to another behavior i' with an *individual* transition rate $\widetilde{w}_a(i',i;t)$. These changes correspond to transitions of the socioconfiguration from $\vec{n}$ to

  $$\vec{n}^a_{i'i} := (n_1^1, \dots, (n_{i'}^a + 1), \dots, (n_i^a - 1), \dots, n_S^A)$$

  with a *configurational* transition rate $w(\vec{n}^a_{i'i},\vec{n};t) = n_i^a \widetilde{w}_a(i',i;t)$, which is proportional to the number n_i^a of individuals who can change the behavior i.

- An individual of subpopulation a may change the behavior from i to i' during a pair interaction with an individual of a subpopulation b who changes the behavior from j to j'. Let transitions of this kind occur with a probability $\widetilde{w}_{ab}(i',j';i,j;t)$ per time unit. The corresponding change of the socioconfiguration from $\vec{n}$ to

  $$\vec{n}^{ab}_{i'j'ij} := (n_1^1, \dots, (n_{i'}^a + 1), \dots, (n_i^a - 1), \dots, (n_{j'}^b + 1), \dots, (n_j^b - 1), \dots, n_S^A)$$

  leads to a configurational transition rate $w(\vec{n}^{ab}_{i'j'ij},\vec{n};t) = n_i^a n_j^b \widetilde{w}_{ab}(i',j';i,j;t)$, which is proportional to the number $n_i^a n_j^b$ of possible pair interactions between individuals of subpopulations a resp. b who show the behavior i resp. j. (Exactly speaking—in order to exclude self-interactions—$n_i^a n_i^a\ \widetilde{w}_{aa}(i',j';i,i;t)$ has to be replaced by $n_i^a(n_i^a - 1)\widetilde{w}_{aa}(i',j';i,i;t)$, if $P(\vec{n},t)$ is not negligible where $n_i^a \gg 1$ does not hold, and $\sum_{j'} \widetilde{w}_{aa}(i',j';i,i;t) \ll \widetilde{w}_a(i',i;t)$ is invalid.)

The resulting configurational transition rate $w(\vec{n}',\vec{n};t)$ is given by

$$w(\vec{n}',\vec{n};t) := \begin{cases} n_i^a \widetilde{w}_a(i',i;t) & \text{if } \vec{n}' = \vec{n}^a_{i'i} \\ n_i^a n_j^b \widetilde{w}_{ab}(i',j';i,j;t) & \text{if } \vec{n}' = \vec{n}^{ab}_{i'j'ij} \\ 0 & \text{otherwise.} \end{cases} \tag{18.3}$$

As a consequence, the explicit form of the master equation (18.2) is

$$\begin{aligned} \frac{d}{dt}P(\vec{n},t) = &\sum_{a,i,i'} \Big[(n_{i'}^a + 1)\widetilde{w}_a(i,i';t)P(\vec{n}^a_{i'i},t) - n_i^a \widetilde{w}_a(i',i;t)P(\vec{n},t)\Big] \\ &+ \frac{1}{2}\sum_{a,i,i'}\sum_{b,j,j'} \Big[(n_{i'}^a + 1)(n_{j'}^b + 1)\widetilde{w}_{ab}(i,j;i',j';t)P(\vec{n}^{ab}_{i'j'ij},t) \\ &\qquad - n_i^a n_j^b \widetilde{w}_{ab}(i',j';i,j;t)P(\vec{n},t)\Big] \end{aligned}$$

(see Helbing (1992a)).

18.3 Most probable behavioral distribution

Because of the great number of possible socioconfigurations $\vec{n}$, the master equation for the determination of the configurational distribution $P(\vec{n},t)$ is usually difficult to solve (even with a computer). However, in many cases one is mainly interested in the *most probable* behavioral distribution

$$P_a(i,t) := \frac{\widehat{n}_i^a(t)}{N_a} .$$

Equations for the most probable occupation numbers $\widehat{n}_i^a(t)$ can be deduced from the LANGEVIN equation

$$\frac{d}{dt}n_i^a(t) \overset{N \gg 1}{=} m_i^a(\vec{n},t) + \textit{fluctuations} , \tag{18.4}$$

which is an approximate reformulation of the master equation (see Helbing (1992)). The *drift coefficients*

$$\begin{aligned} m_i^a(\vec{n},t) &:= \textstyle\sum_{\boldsymbol{n}'} (n_i'^a - n_i^a) w(\vec{n}',\vec{n};t) \\ &= \textstyle\sum_{i'} \Big[\overline{w}^a(i,i';t)n_{i'}^a - \overline{w}^a(i',i;t)n_i^a\Big] , \end{aligned} \tag{18.5}$$

where

$$\overline{w}^a(i',i;t) := \widetilde{w}_a(i',i;t) + \sum_b \sum_{j'} \sum_j \widetilde{w}_{ab}(i',j';i,j;t)n_j^b \tag{18.6}$$

have the meaning of *effective transition rates*, describe the temporal drift of the configurational distribution $P(\vec{n},t)$ (see Helbing (1992, 1992a)). Obviously, the contributions $\widetilde{w}_{ab}(i',j';i,j;t)n_j^b$ due to pair interactions are proportional to the number n_j^b of possible interaction partners.

The LANGEVIN equation (18.4) determines the behavior of the socioconfiguration $\vec{n}(t)$ in dependence of process immanent fluctuations. As a consequence,

$$\frac{d}{dt}\widehat{n}_i^a(t) \overset{N \gg 1}{=} m_i^a(\widehat{\vec{n}}, t) \qquad (18.7)$$

are the equations for the most probable occupation numbers $\widehat{n}_i^a(t)$. A measure for the reliability (or representativity) of $\widehat{n}_i^a(t)$ with respect to the possible temporal developments of $n_i^a(t)$ are the (co)variances of $n_i^a(t)$ (see Helbing (1992, 1992b)).

18.4 Kinds of pair interactions

The pair interactions

$$i', j' \longleftarrow i, j$$

of two individuals of subpopulations a resp. b who change their behavior from i resp. j to i' resp. j' can be completely classified according to the following scheme:

$$\left.\begin{array}{lll} i,i & \longleftarrow & i,i \\ i,j & \longleftarrow & i,j \end{array}\right\} (0)$$

$$\left.\begin{array}{llll} i,i & \longleftarrow & i,j & (i \neq j) \\ j,j & \longleftarrow & i,j & (i \neq j) \end{array}\right\} (1)$$

$$\left.\begin{array}{llll} i,j' & \longleftarrow & i,i & (j' \neq i) \\ i',j & \longleftarrow & j,j & (i' \neq j) \\ i',j' & \longleftarrow & i,i & (i' \neq i, j' \neq i) \end{array}\right\} (2)$$

$$\left.\begin{array}{llll} i,j' & \longleftarrow & i,j & (i \neq j, j' \neq j, j' \neq i) \\ i',j & \longleftarrow & i,j & (i \neq j, i' \neq i, i' \neq j) \\ i',j' & \longleftarrow & i,j & (i \neq j, i' \neq i, j' \neq j, i' \neq j, j' \neq i) \end{array}\right\} (3)$$

$$\left.\begin{array}{llll} j,i & \longleftarrow & i,j & (i \neq j) \\ i',i & \longleftarrow & i,j & (i \neq j, i' \neq i, i' \neq j) \\ j,j' & \longleftarrow & i,j & (i \neq j, j' \neq j, j' \neq i) \end{array}\right\} (4)$$

Obviously, the interpretation of the above *kinds* $k \in \{0, 1, \dots, 4\}$ of pair interactions is the following:

(0) During interactions of kind (0) both individuals do not change their behavior. These interactions can be omitted in the following, since they have no contribution to the change of $P(\vec{n}, t)$ or $n_i^a(t)$.

(1) The interactions (1) describe *imitative processes* (processes of persuasion), i.e., the tendency to take over the behavior of another individual.

(2) The interactions (2) describe *avoidance processes*, where an individual changes the behavior when meeting another individual showing the same behavior. Processes of this kind are known as aversive behavior, defiant behavior or snob effect.

(3) The interactions (3) represent some kind of *compromising processes*, where an individual changes the behavior to a new one (the "compromise") when meeting an individual with another behavior. Such processes are found, if a certain behavior cannot be maintained when confronted with another behavior.

(4) The interactions (4) describe imitative processes, in which an individual changes the behavior despite of the fact, that he or she convinces the interaction partner of his resp. her behavior. Processes of this kind are very improbable and shall be excluded in the following discussion.

For the transition rates corresponding to these kinds of interaction processes the following plausible form shall be assumed (see Helbing (1992)):

$$\widetilde{w}_{ab}(i', j'; i, j; t) := \widetilde{\nu}_{ab}(t) \cdot \begin{cases} p_{ab}^1(i'|i;t) & \text{if } i' = j \text{ and } j' = j \\ p_{ba}^1(j'|j;t) & \text{if } j' = i \text{ and } i' = i \\ 0 & \text{if } i' = j \text{ and } j' \neq j \\ 0 & \text{if } j' = i \text{ and } i' \neq i \\ p_{ab}^k(i'|i;t) p_{ba}^k(j'|j;t) & \text{otherwise } (k \in \{2,3\}). \end{cases} \qquad (18.8)$$

Here,

$$\nu_{ab}(t) := N_b \widetilde{\nu}_{ab}(t)$$

is the *contact rate* between an individual of subpopulation a with individuals of subpopulation b. $p_{ab}^k(j|i;t)$ is the probability of an individual of subpopulation a to change the behavior from i to j during a pair interaction of kind k with an individual of subpopulation b, i.e.,

$$\sum_j p_{ab}^k(j|i;t) = 1 .$$

Let us assume

$$p_{ab}^k(j|i;t) := f_{ab}^k(t) R_a(j, i; t) ,$$

where $f_{ab}^k(t)$ is a measure for the *frequency* of pair interactions of kind k between individuals of subpopulation a and b, and $R_a(j, i; t)$ is a measure for the *readiness* of individuals belonging to subpopulation a to change the behavior from i to j during a pair interaction. Inserting the rate (18.8) of pair interactions into (18.6) and using the conventions

$$\begin{aligned} w_a(i', i; t) &:= \widetilde{w}_a(i', i; t) , \\ w_{ab}(i', j'; i, j; t) &:= N_b \widetilde{w}_{ab}(i', j'; i, j; t) , \\ \nu_{ab}^k(t) &:= \nu_{ab}(t) f_{ab}^k(t) , \end{aligned}$$

we arrive at the equations

$$\frac{d}{dt} P_a(i, t) = \sum_{i'} \Big[w^a(i, i'; t) P_a(i', t) - w^a(i', i; t) P_a(i, t) \Big] \qquad (18.9)$$

for the most probable behavioral distribution (see (18.7), (18.5), (18.6)), where

$$w^a(i, i'; t) := w_a(i, i'; t) + \nu_a(i, i'; t) R_a(i, i'; t) \qquad (18.10)$$

with

$$\nu_a(i,i';t) := \sum_b \Big[\big(\nu^1_{ab}(t) - \nu^3_{ab}(t)\big) P_b(i,t) + \big(\nu^2_{ab}(t) - \nu^3_{ab}(t)\big) P_b(i',t) + \nu^3_{ab}(t) \Big] \quad (18.11)$$

are *effective* transition rates (see Helbing (1992, 1991a)). The effective transition rates include contributions of spontaneous behavioral changes, and of behavioral changes due to pair interactions (i.e., of imitative, avoidance and compromising processes). (18.9), (18.11) are BOLTZMANN-*like equations* (see Boltzmann (1964), Helbing (1992a)).

Due to (18.1), (18.3), and $0 \le n^a_i \le N_a$ we have the relations

$$\sum_i P_a(i,t) = 1 \qquad \text{and} \qquad 0 \le P_a(i,t) \le 1 \,.$$

Therefore, $P_a(i,t)$ can be interpreted as the *(most probable) fraction* of individuals within subpopulation a who show the behavior i. With respect to the *total* population, the fraction $P(i,t)$ of individuals with behavior i is given by

$$P(i,t) = \frac{\widehat{n}_i}{N} := \frac{\sum_a \widehat{n}^a_i}{N} = \sum_a \frac{N_a}{N}\frac{\widehat{n}^a_i}{N_a} = \sum_a \frac{N_a}{N} P_a(i,t) \,.$$

18.4.1 Computer simulations

For an illustration of the BOLTZMANN-like equations (18.9), (18.11) we shall assume to have two subpopulations ($A = 2$), and three different behaviors ($S = 3$). With

$$R_a(i',i;t) := \frac{e^{U_a(i',t)-U_a(i,t)}}{D_a(i',i;t)} \,, \quad (18.12)$$

(see Weidlich and Haag (1988), Helbing (1992)) the readiness $R_a(i',i;t)$ for an individual of subpopulation a to change the attitude from i to i' will be the greater, the greater the difference of the *utilities* $U_a(.,t)$ of behaviors i' and i is, and the smaller the *incompatibility ("distance")*

$$D_a(i',i;t) = D_a(i,i';t) > 0$$

between the behaviors i and i' is.

In the following computer simulations $D_a(i',i;t) \equiv 1$ has been taken. For both subpopulations the *prefered* behavior, i.e., the behavior with the *greatest* utility $U_a(i,t)$ is represented by a solid line, whereas the behavior with the lowest utility is represented by a dotted line, and the behavior with medium utility by a broken line. Figures 18.1–18.6 show the effects of imitative processes ($\nu^1_{ab}(t) \equiv 1$, $\nu^2_{ab}(t) \equiv 0 \equiv \nu^3_{ab}(t)$), of avoidance processes ($\nu^2_{ab}(t) \equiv 1$, $\nu^1_{ab}(t) \equiv 0 \equiv \nu^3_{ab}(t)$), resp. of compromising and imitative processes ($\nu^3_{ab}(t) \equiv 1 \equiv \nu^1_{ab}(t)$, $\nu^2_{ab}(t) \equiv 0$)

a) for *equal* behavioral preferenes ($U_1(1) = c = U_2(1)$, $U_1(2) = 0 = U_2(2)$, $U_1(3) = -c = U_2(3)$), and

b) for *different* behavioral preferences ($U_1(1) = c = U_2(2)$, $U_1(2) = 0 = U_2(1)$, $U_1(3) = -c = U_2(3)$).

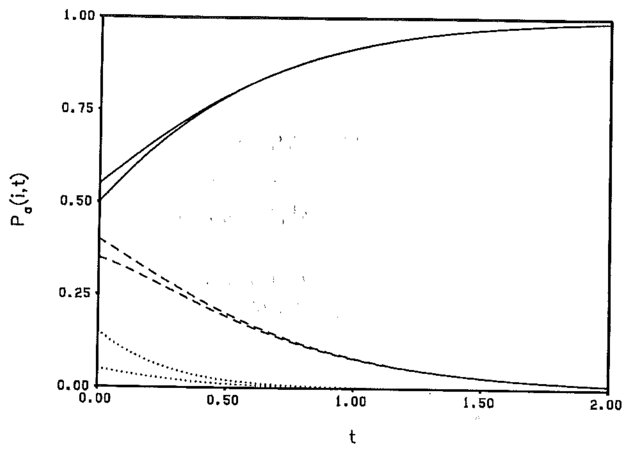

Figure 18.1. Effect of imitative processes for two subpopulations prefering the same behavior ($c = 0.5$): Only the fraction of the prefered behavior (—) is increasing. The other behaviors vanish in the course of time.

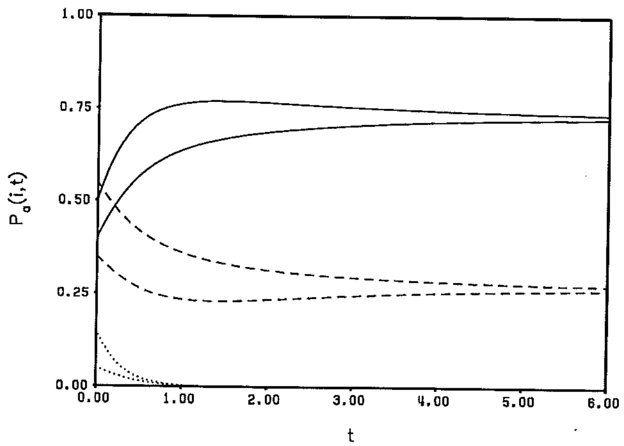

Figure 18.2. Effect of imitative processes for two subpopulations prefering different behaviors ($c = 0.5$): The prefered behavior (—) becomes the predominating one in each subpopulation, but the behavior which is prefered in the *other* subpopulation (– –) can also convince a certain fraction of individuals. A behavior which is not prefered by *any* subpopulation (···) vanishes.

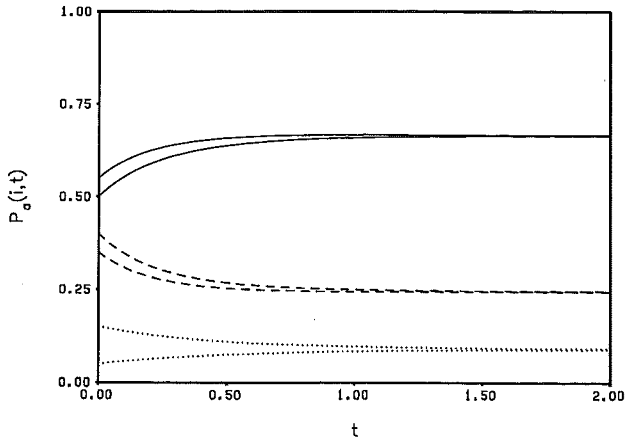

Figure 18.3. Effect of avoidance processes for two subpopulations prefering the same behavior ($c = 1$): The fraction of the prefered behavior (—) is limited, since the subpopulations avoid to show the same behavior. As a consequence, the other behaviors are also used by a certain fraction of individuals.

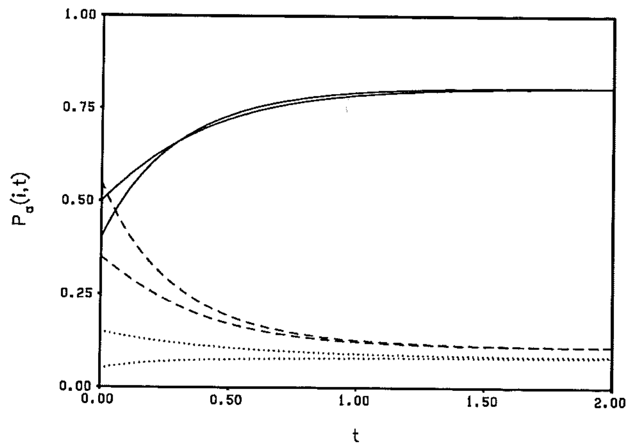

Figure 18.4. Effect of avoidance processes for two subpopulations prefering different behaviors ($c = 1$): The fraction of the prefered behavior (—) wins a greater majority in comparison with figure 18.3, since the situations of avoidance are reduced.

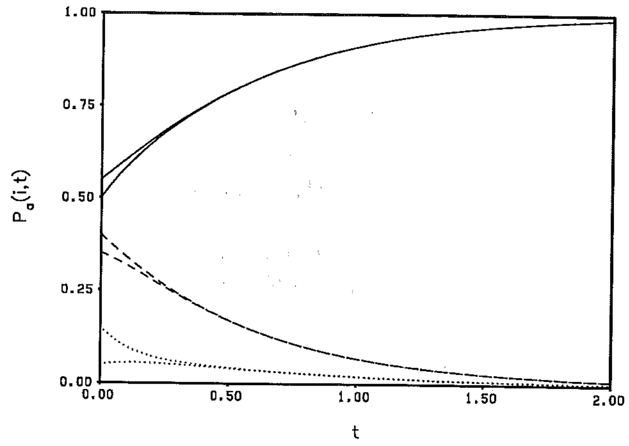

Figure 18.5. Effect of compromising and imitative processes for two subpopulations prefering the same behavior ($c = 0.5$): Only the prefered behavior (—) survives, since a readiness for compromises is not necessary.

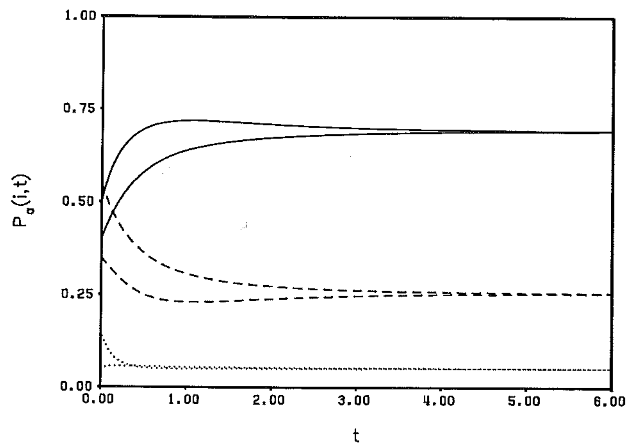

Figure 18.6. Effect of compromising and imitative processes for two subpopulations prefering different behaviors ($c = 0.5$): Most of the individuals show the prefered behavior (—), but a certain fraction of individuals also decides for a compromise ($\cdots$).

18.5 Game dynamical equations

In game theory, i denotes a (behavioral) *strategy*. Let $E_a(i,t)$ be the *expected success* of a strategy i for an individual of subpopulation a, and

$$\langle E_a \rangle := \sum_i E_a(i,t) P_a(i,t)$$

the *mean expected success*. If the *relative* increase

$$\frac{dP_a(i,t)/dt}{P_a(i,t)}$$

of the fraction $P_a(i,t)$ is assumed to be proportional to the difference $[E_a(i,t) - \langle E_a \rangle]$ between the expected and the mean expected success, one obtains the *game dynamical equations*

$$\frac{d}{dt} P_a(i,t) = \nu_a(t) P_a(i,t) \Big[E_a(i,t) - \langle E_a \rangle \Big] . \qquad (18.13)$$

That means, the fractions of strategies with an expected success that exceeds the average $\langle E_a \rangle$ are growing, whereas the fractions of the remaining strategies are falling. For the expected success $E_a(i,t)$, one often takes the form

$$E_a(i,t) := \sum_b \sum_j A_{ab}(i,j;t) P_b(j,t) , \qquad (18.14)$$

where $A_{ab}(i,j;t)$ have the meaning of *payoffs*. We shall assume

$$A_{ab}(i,j;t) := r_{ab}(t) E_{ab}(i,j;t) \quad \text{with} \quad r_{ab}(t) := \frac{\nu_{ab}(t)}{\sum_c \nu_{ac}(t)} ,$$

where $r_{ab}(t)$ is the *relative contact rate* of an individual of subpopulation a with individuals of subpopulation b, and $E_{ab}(i,j;t)$ is the *success* of strategy i for an individual of subpopulation a during an interaction with an individual of subpopulation b who uses strategy j. Since $r_{ab}(t)P_b(j,t)$ is the relative contact rate of an individual of subpopulation a with individuals of subpopulation b who use strategy j, $E_a(i,t)$ is the mean (or *expected*) success of strategy i for an individual of subpopulation a in interactions with other individuals.

By inserting (18.14) and

$$\langle E_a \rangle = \sum_{i'} \sum_{b,j} P_a(i',t) A_{ab}(i',j;t) P_b(j,t)$$

into (18.13), one obtains the explicit form

$$\frac{d}{dt} P_a(i,t) = \nu_a(t) P_a(i,t) \left[\sum_{b,j} A_{ab}(i,j;t) P_b(j,t) - \sum_{i'} \sum_{b,j} P_a(i',t) A_{ab}(i',j;t) P_b(j,t) \right] \qquad (18.15)$$

of the game dynamical equations. (18.15) is a *continuous* formulation of game theory (see Hofbauer and Sigmund (1988)). Equations of this kind are very useful for the investigation and understanding of the competition or cooperation of individuals (see, e.g., Mueller (1990), Hofbauer and Sigmund (1988), Schuster et. al. (1981)).

A slightly generalized form of (18.13),

$$\frac{d}{dt} P_a(i,t) = \sum_{i'} \Big[w_a(i,i';t) P_a(i',t) - w_a(i',i;t) P_a(i,t) \Big] \qquad (18.16)$$

$$+ \nu_a(t) P_a(i,t) \Big[E_a(i,t) - \langle E_a \rangle \Big] , \qquad (18.17)$$

is also known as *selection mutation equation* (Hofbauer and Sigmund (1988)): (18.17) can be understood as effect of a *selection* (if $E_a(i,t)$ is interpreted as *fitness* of strategy i), and (18.16) can be understood as effect of *mutations*. Equation (18.16), (18.17) is a powerful tool in evolutionary biology (see Eigen (1971), Fisher (1930), Eigen and Schuster (1979), Hofbauer and Sigmund (1988), Feistel and Ebeling (1989)). In game theory, the mutation term could be used for the description of *trial and error* behavior or of accidental variations of the strategy.

18.5.1 Connection between BOLTZMANN-like and game dynamical equations

One expects that there *must* be a connection between the BOLTZMANN-like equations (18.9), (18.11) and the game dynamical equations (18.16), (18.17), since they are both quantitative models for behavioral changes. A comparison of (18.9), (18.11) with (18.16), (18.17) shows, that both models can become identical only under the conditions

$$\nu^1_{ab}(t) = \nu_a(t) \delta_{ab} , \qquad \nu^2_{ab}(t) = 0 , \qquad \nu^3_{ab}(t) = 0 . \qquad (18.18)$$

That means, the game dynamical equations include spontaneous and imitative behavioral changes, but they exlude avoidance and compromising processes.

In order to make the analogy between the game dynamical and the BOLTZMANN-like equations complete the following assumptions have to be made:

- In interactions with other individuals the expected success

$$E_a(i,t) = \sum_{b,j} \frac{\nu_{ab}(t)}{\sum_c \nu_{ac}(t)} E_{ab}(i,j;t) P_b(j,t) \qquad (18.19)$$

  of a strategy is evaluated. This is possible, since an individual is able to determine the quantities $\nu_{ab}(t)$, $P_b(j,t)$ and $E_{ab}(i,j;t)$: An individual of subpopulation a meets individuals of subpopulation b with a contact rate of $\nu_{ab}(t)$. With a probability of $P_b(j,t)$, the individuals of subpopulation b use the strategy j. During interactions with individuals of subpopulation b who use the strategy j, an individual of subpopulation a has a success of $E_{ab}(i,j;t)$ if using the strategy i.

- In interactions with individuals of the *same* subpopulation an individual tends to take over the strategy of another individual, if the expected success would increase: If an individual who uses a strategy i meets another individual of the same subpopulation who uses a strategy j, they will compare their expected success' $E_a(i,t)$ resp. $E_a(j,t)$ (by observation or exchange of their experiences). The individual with strategy i will imitate the other's strategy j with a probability $p^1_{ab}(j|i;t)$ that is growing with the expected increase

$$\Delta_{ji}E_a := E_a(j,t) - E_a(i,t)$$

of success. If a change of strategy would imply a decrease of success ($\Delta_{ji}E_a < 0$), the individual will not change the strategy i. Therefore, the readiness for replacing the strategy i by j during an interaction within the same subpopulation can be assumed to be

$$R_a(j,i;t) := \max\Big(E_a(j,t) - E_a(i,t), 0\Big)\,, \qquad (18.20)$$

where $\max(x,y)$ is the maximum of the two numbers x and y. However, due to different criteria for the grade of success, the expected success of a strategy i will usually be varying with the subpopulation a (i.e., $E_a(i,t) \neq E_b(i,t)$ for $a \neq b$). As a consequence, an imitative behavior of individuals belonging to *different* subpopulations is not plausible, and we shall assume

$$f^1_{ab}(t) := \delta_{ab}\,, \qquad \text{i.e.,} \qquad \nu^1_{ab}(t) = \nu_{aa}(t)\delta_{ab}\,.$$

Inserting (18.18), (18.19) and (18.20) into the BOLTZMANN-like equations (18.9), (18.11), the game dynamical equations (18.16), (18.17) result as a special case, since

$$\max\Big(E_a(i,t) - E_a(j,t), 0\Big) - \max\Big(E_a(j,t) - E_a(i,t), 0\Big) = E_a(i,t) - E_a(j,t)\,.$$

18.5.2 Stochastic version of the game dynamical equations

Applying the formalism of section 18.2, a stochastic version of the game dynamical equations can easily be formulated. This is given by the master equation (18.2) with the configurational transition rates (18.3) and

$$\begin{aligned} w_{ab}(i',j';i,j;t) &:= N_b\widetilde{w}^1_{ab}(i',j';i,j;t) \\ &:= \nu_a(t)\delta_{ab}\widehat{R}_a(i,j;t)\delta_{ii'}\delta_{ij'}(1-\delta_{ij}) \\ &+ \nu_a(t)\delta_{ab}\widehat{R}_a(j,i;t)\delta_{jj'}\delta_{ji'}(1-\delta_{ij})\,, \end{aligned}$$

where

$$\widehat{R}_a(j,i;t) := \max\Big(\widehat{E}_a(j,t) - \widehat{E}_a(i,t), 0\Big)$$

and

$$\widehat{E}_a(i,t) := \sum_b \sum_j A_{ab}(i,j;t)\frac{n^b_j}{N_b}$$

(compare to Feistel and Ebeling (1989), Ebeling and Feistel (1982), Ebeling et. al. (1990)). A comparison with (18.7), (18.9), (18.11) shows, that the ordinary game dynamical equations (18.16), (18.17) are equations for the most probable behavioral distribution.

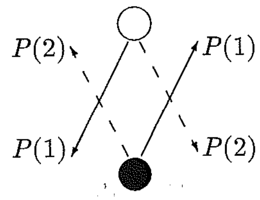

Figure 18.7. For pedestrians with an opposite direction of motion it is advantageous, if both prefer either the right hand side or the left hand side when trying to pass each other. Otherwise, they would have to stop in order to avoid a collision. The probability $P(1)$ for choosing the right hand side is usually greater than the probability $P(2) = 1 - P(1)$ for choosing the left hand side.

18.5.3 Selforganization of behavioral conventions by competition between equivalent strategies

This section gives an illustration of the methods and results derived in sections 18.5 and 18.5.2. As an example, we shall consider a case with one subpopulation only ($A = 1$), and, therefore, omit the index a in the following. Let us suppose the individuals to choose between two equivalent strategies $i \in \{1,2\}$, i.e., the payoff matrix $\underline{A}(t)$ shall be symmetrical:

$$\underline{A}(t) \equiv \Big(A(i,j;t)\Big) := \begin{pmatrix} A+B & B \\ B & A+B \end{pmatrix}. \qquad (18.21)$$

According to the relation

$$n_1 + n_2 = N$$

(see (18.1)), the fraction $P(2,t) = 1 - P(1,t)$ is already determined by $P(1,t)$. By scaling the time,

$$\nu(t) \equiv 1$$

can be presupposed. For the spontaneous change of strategies due to trial and error we shall assume the transition rates

$$w(j,i;t) := W\,. \qquad (18.22)$$

A situation of the above kind is the avoidance behavior of pedestrians (see Helbing (1991)): In pedestrian crowds with two opposite directions of movement, the pedestrians have sometimes to avoid each other in order to exclude a collision. For an avoidance maneuver to be successful, both pedestrians concerned have to pass the respective other pedestrian either on the right hand side or on the left hand side. Otherwise, both pedestrians have to stop (see figure 18.7). Therefore, both strategies (to pass pedestrians on the right hand side or to pass them on the left hand side) are equivalent, but the success of a strategy grows with the number n_i of individuals who use the *same* strategy. In the payoff matrix (18.21) we have $A > 0$, then.

The game dynamical equations (18.16), (18.17) corresponding to (18.21), (18.22) have the explicit form

$$\frac{d}{dt}P(i,t) = -2\left(P(i,t) - \frac{1}{2}\right)\Big[W + AP(i,t)\Big(P(i,t) - 1\Big)\Big]\,. \qquad (18.23)$$

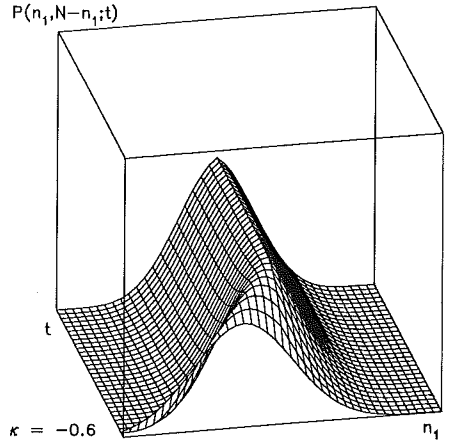

Figure 18.8. Probability distribution $P(\vec{n},t) \equiv P(n_1, N-n_1; t)$ of the socioconfiguration $\vec{n} = (n_1, N-n_1)$ for two equivalent competing strategies. Mutation dominated region ($\kappa < 0$): Since $P(n_1, N-n_1; t)$ has, after a certain time interval, one maximimum at $n_1 = N/2$, each strategy will most probably be used by about one half of the individuals.

According to (18.23), $P(i) = 1/2$ is a stationary solution. This solution is stable only for

$$\kappa := 1 - \frac{4W}{A} < 0\,,$$

i.e., if spontaneous strategy changes due to trial and error (the "mutations") are dominating. For $\kappa > 0$ the stationary solution $P(i) = 1/2$ is unstable, and the game dynamical equations (18.23) can be rewritten in the form

$$\frac{d}{dt}P(i,t) = -2\left(P(i,t) - \frac{1}{2}\right)\left(P(i,t) - \frac{1+\sqrt{\kappa}}{2}\right)\left(P(i,t) - \frac{1-\sqrt{\kappa}}{2}\right).$$

That means, for $\kappa > 0$ we have two additional stationary solutions $P(i) = (1+\sqrt{\kappa})/2$ and $P(i) = (1-\sqrt{\kappa})/2$, which are stable. Depending on the random initial condition $P(i, t_0)$, one strategy will win a majority of $100 \cdot \sqrt{\kappa}$ percent. This majority is the greater, the smaller the rate W of spontaneous strategy changes is.

At the *critical point* $\kappa = \kappa_0 := 0$ there appears a *phase transition.* This can be seen best in figures 18.8–18.9, where the distribution $P(\vec{n},t) \equiv P(n_1,n_2;t) = P(n_1, N-n_1; t)$ loses its unimodal form for $\kappa > 0$. As a consequence of the phase transition, one strategy is prefered, i.e. a behavioral convention develops.

The crease of $P(n_1, N-n_1; t)$ at $n_1 = N/2 = n_2$ is a result of the crease of the function $\widehat{R}_a(j,i;t) = \max(\widehat{E}_a(j,t) - \widehat{E}_a(i,t), 0)$. It can be avoided by using the modified *ansatz*

$$\widehat{R}_a(j,i;t) := \frac{e^{\widehat{E}_a(j,t) - \widehat{E}_a(i,t)}}{D_a(j,i;t)}$$

(compare to (18.12)), which also shows a phase transition for $\kappa = 0$ (see figure 18.11).

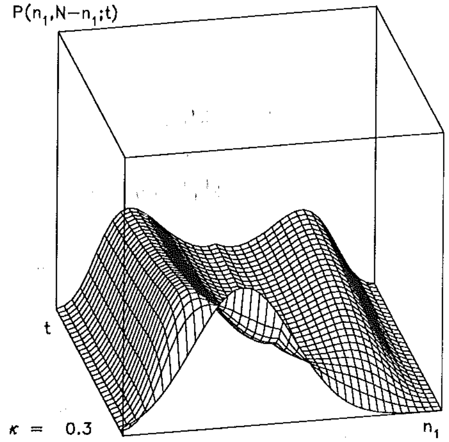

Figure 18.9. As figure 18.8, but after the phase transition ($\kappa > 0$): The configurational distribution $P(n_1, N-n_1; t)$ becomes multimodal with maxima that are symmetrical with respect to $N/2$, because of the equivalence of the strategies. Due to the maxima at $n_1 > N/2$ and $n_2 = N - n_1 > N/2$, one of the strategies will very probably win a majority of users. This implies the selforganization of a behavioral convention.

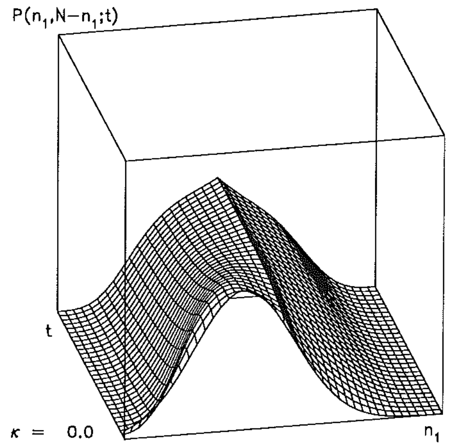

Figure 18.10. As figure 18.8, but for the critical point $\kappa = 0$: The broadness of the probability distribution $P(n_1, N-n_1; t)$ indicates *critical fluctuations*, i.e. a phase transition.

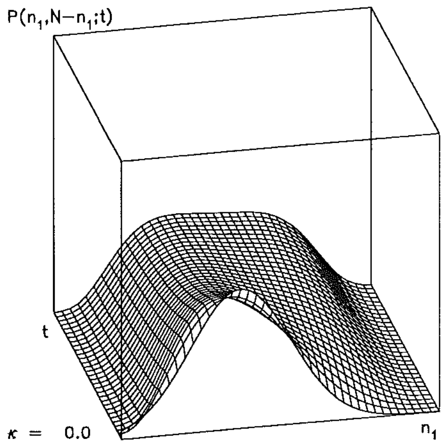

Figure 18.11. As figure 18.10, but with a modified *ansatz* for the readiness $R_a(j, i; t)$ to change the behavior from i to j, which does not produce a crease of $P(n_1, N - n_1; t)$ at $N/2$.

18.6 Summary and Conclusions

A quite general model for behavioral changes has been developed, which takes into account spontaneous changes and changes due to pair interactions. Three kinds of pair interactions have been distinguished: imitative, avoidance and compromising processes. The game dynamical equations result for a special case of imitative processes. They can be interpreted as equations for the most probable behavioral distribution and allow the description of social selforganization.

Acknowledgements

The author wants to thank Prof. Dr. W. Weidlich and PD Dr. G. Haag for inspiring discussions.

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