ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

851-0585-04L – Modeling and Simulating Social Systems with MATLAB

Lecture 7 – Continuous Simulations

Karsten Donnay and Stefano Balietti

Chair of Sociology, in particular of

Modeling and Simulation

Schedule of the course

Introduction to MATLAB

21.02.
28.02.

07.03.
14.03.
21.03.
28.03.
04.04.
18.04.
02.05.

Introduction to social-science modeling and simulation

Working on projects (seminar theses)

09.05.
16.05.
23.05.
30.05.

Handing in seminar thesis and giving a presentation
(final deadlines to be communicated)

Schedule of the course

Introduction to MATLAB

21.02.

28.02.

Different ways of Representing space

07.03. → **Dynamical Systems (no-space)**

14.03. → **Cellular Automata (grid)**

Working on projects (seminar theses)

21.03.

28.03. → **Networks (graphs)**

04.04. → **Continuous Space**

18.04.

02.05.

09.05.

16.05.

23.05.

30.05.

Handing in seminar thesis and giving a presentation
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Goals of Lecture 6: students will

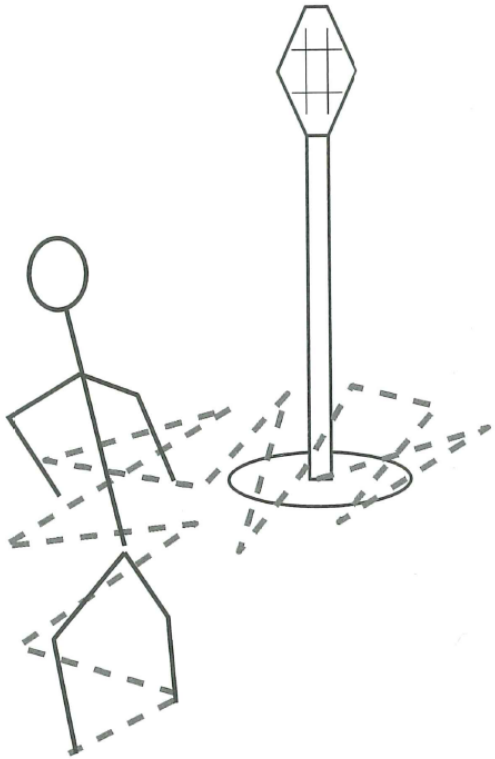
1. Consolidate knowledge acquired during lecture 6, through *brief* repetition of the main points.
2. Receive an *essential*, but *commensurate* introduction about Stochastic Processes.
3. Discover the **Micro-Macro link** in continuous modeling, bridging Random Walk and Diffusion Equation.
4. Understand the *basics* of **Motion Dynamics** and learn how to simulate processes like Brownian Motion in Continuum Space.
5. Have a closer glance into two examples of Pedestrian Simulation, confronting grids with continuum space.

Repetition

- A **Graph** is defined as ***G = (N,L)***, where ***N*** is the number of nodes and ***L*** their links.
- A Graph is enriched with extra information, encoded either in the nodes or in the links, is called **Network**.
- A Graph can be mathematically described at the level of:
  - Nodes: degree, betweeness centrality,…
  - Graph: degree distribution, connectedness, giant component
- Network **topologies**: *Random*, *Scale Free*, *Small World*
- **Generators**: *Erdos-Renyi*, *Preferential attachment*, *Self-similar Generators.*

All began with the Drunkard's Walk

- The drunkard takes a series of steps away from the lamp post, each with a random angle.



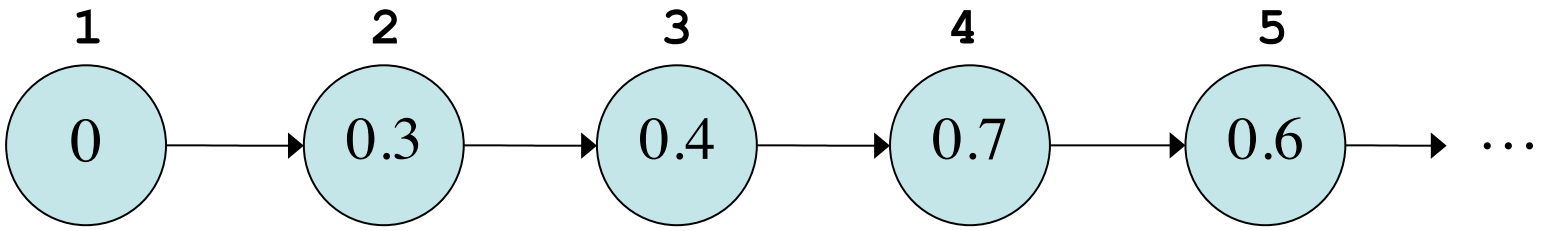
(Figure Resource: Sethna JP, 2006)

Stochastic Process

- A stochastic process:

$$(X_t)_{t \in I}$$

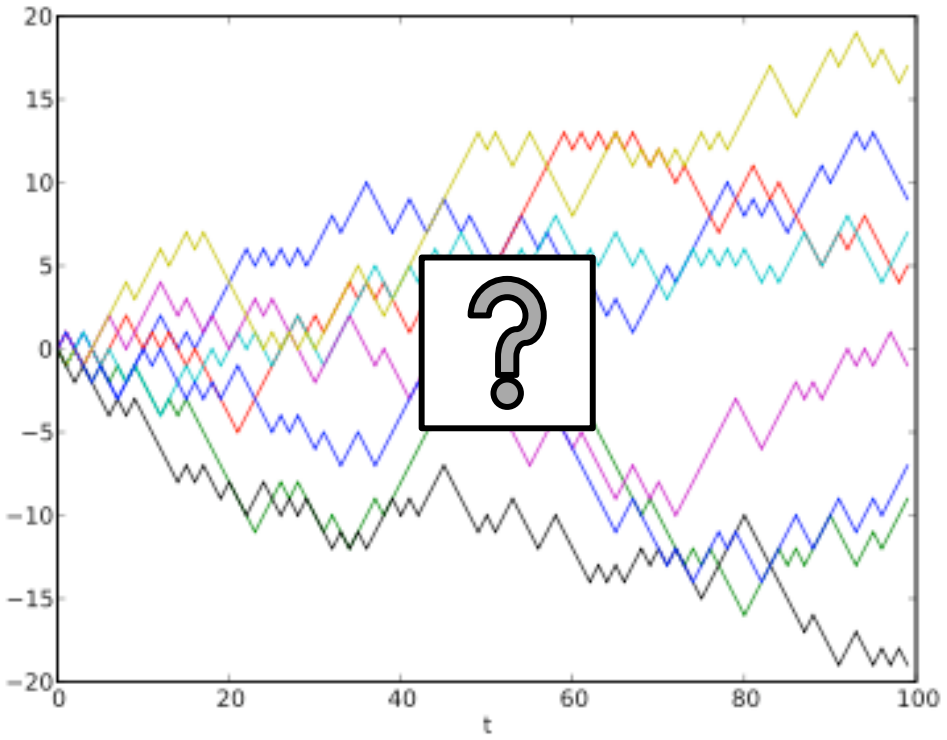
is a family of random variables indexed on an interval I , where $I \subset R$.



- Intuitively, a stochastic process is an infinite-dimension random vector

Stochastic Process

- In reality, we observe just a **single realization** of such process, among all the possible ones.
- E.g. Stock Market Boom vs Stock Market Slump



White Noise (WN)

- **White Noise** is a stochastic process whose *t-th* element has the following properties:
  - $E(x_t) = 0$
  - $E(x_t^2) = \sigma^2$
  - $\gamma_k = 0$ for $|k| > 0$ (autocorrelation coeff)

- That means that the white noise is succession of uncorrelated random variables with zero mean and constant variance.

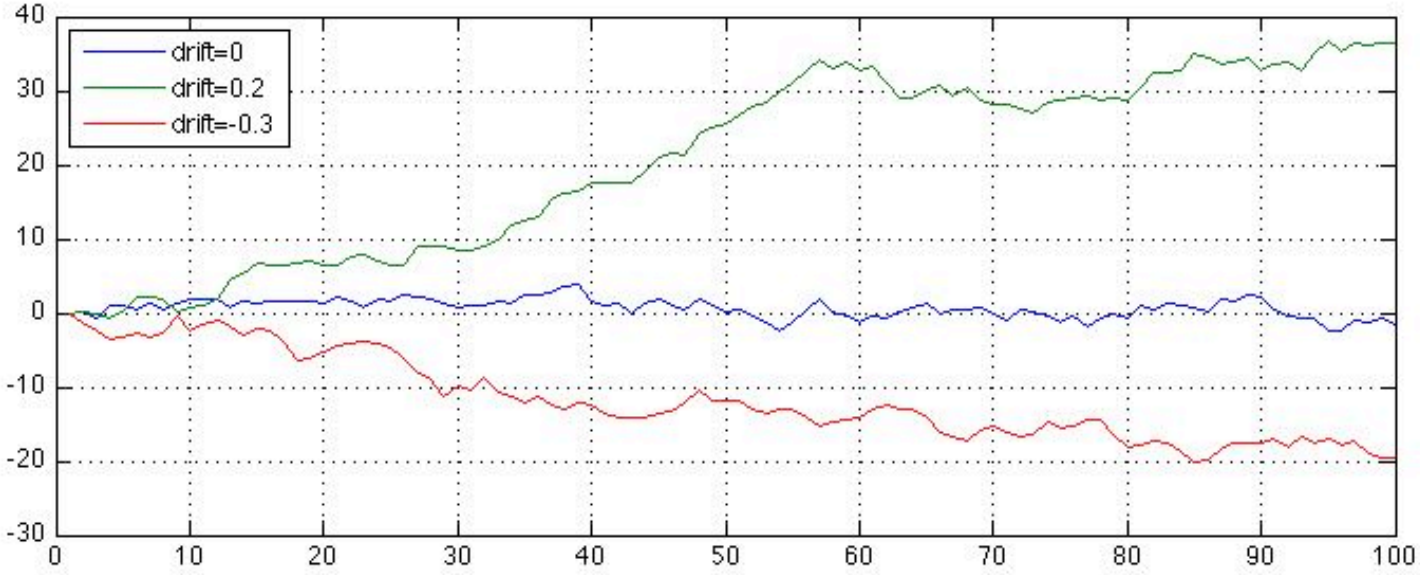
Random Walk (RW)

- **Random Walk** is a stochastic process whose *t-th* element is determined as follows:

$$X_t = \mu + X_{t-1} + \epsilon_t$$

- μ is the drift, constantly leading the process towards its sign
- ϵ_t represents the random increments.

Random Walk (RW)




```
function rw = rw(steps,drift)
    rw = zeros(steps,1);
    for i=2:steps
        rw(i) = drift + rw(i-1) + randn(1,1);
    end
end
```

RW: on the random increments ϵ_t

- **RW1.** ϵ_t = IID (0, σ^2). Independent and Identically Distributed.
  - classical hypothesis -> computational simplification
- **RW2**. ϵ_t are still independent, but no more identically distributed.
  - Different density functions across times are possible
- **RW3.** ϵ_t are simply uncorrelated and basically follow a WN process. This is the weakest RW form.
  - Admits non linear correlations among past values.

Random Walk and Diffusion Equation

- Consider a general, uncorrelated random walk where at each time step Δt the particle's position $x(t)$ changes by a step $l(t)$:

$$x(t+\Delta t) = x(t) + l(t).$$

- The probability distribution for each step is $D(l)$, which has zero mean and standard deviation α.

From Random Walk to Diffusion Equation

- For the particle to go from x' at time t to x at time $t+\Delta t$, the step $l(t) = x - x'$. Such event occurs with probability $D(x-x')$ times the probability density $\rho(x',t)$. Therefore, we have

$$\rho(x,t+\Delta t) = \int_{-\infty}^{+\infty} \rho(x',t) D(x-x') dx'$$

$$= \int_{-\infty}^{+\infty} \rho(x-z,t) D(z) dz$$

$$z = x - x'$$

The Micro-Macro Link

- Using Taylor expansion, we have

$$\rho(x,t+\Delta t) \approx \int [\rho(x,t) - z\frac{\partial \rho}{\partial x} + \frac{z^2}{2}\frac{\partial^2 \rho}{\partial x^2}]D(z)dz$$

$$= \rho(x,t)\int D(z)dz - \frac{\partial \rho}{\partial x}\int zD(z)dz + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\int z^2 D(z)dz$$

$$= \rho(x,t) + \frac{1}{2}\frac{\partial^2 \rho}{\partial x^2}\alpha^2$$

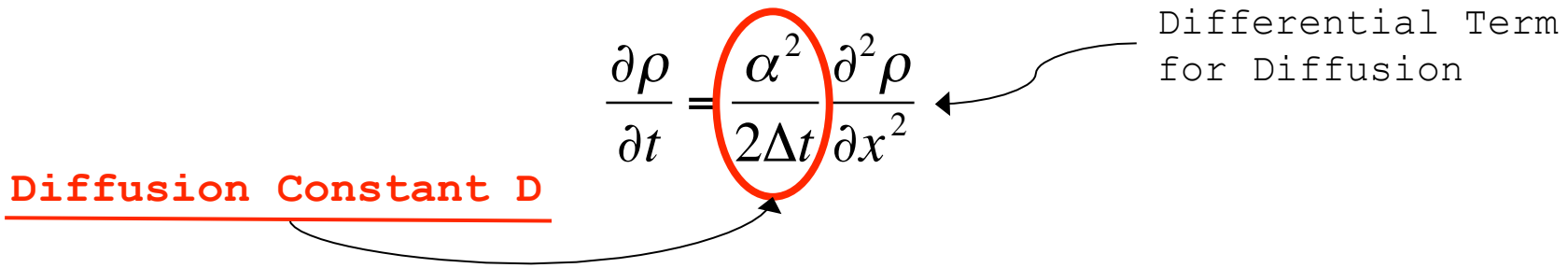
$$\rho(x,t+\Delta t) \approx \rho(x,t) + \frac{\partial \rho}{\partial t}\Delta t$$

The following is the diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{\alpha^2}{2\Delta t}\frac{\partial^2 \rho}{\partial x^2}$$

The Micro-Macro Link: Take Home Msg

- Diffusion:



- Constant drift of the ensemble with the characteristic constant D given by the properties of RW
- A simple and controllable process such as RW applied to single agents can give rise to seemingly organized macroscopic behavior of the ensemble.

The Taylor Series

- The **Taylor series** is an approximation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point.
- F(x) can thus be rewritten as:
  - $f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots.$
  - $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
    - a = single point

The Taylor Series in Matlab

- What's the Taylor Series for $f(x) = e^x$?

```
>> syms x
>> f = exp(x);
>> taylor(f)

ans =

x^5/120 + x^4/24 + x^3/6 + x^2/2 + x + 1
```

The Taylor Series in Matlab

- What's the Taylor Series for $f(x) = e^x$?

```
>> syms x
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>> taylor(f)

ans =

x^5/120 + x^4/24 + x^3/6 + x^2/2 + x + 1
```
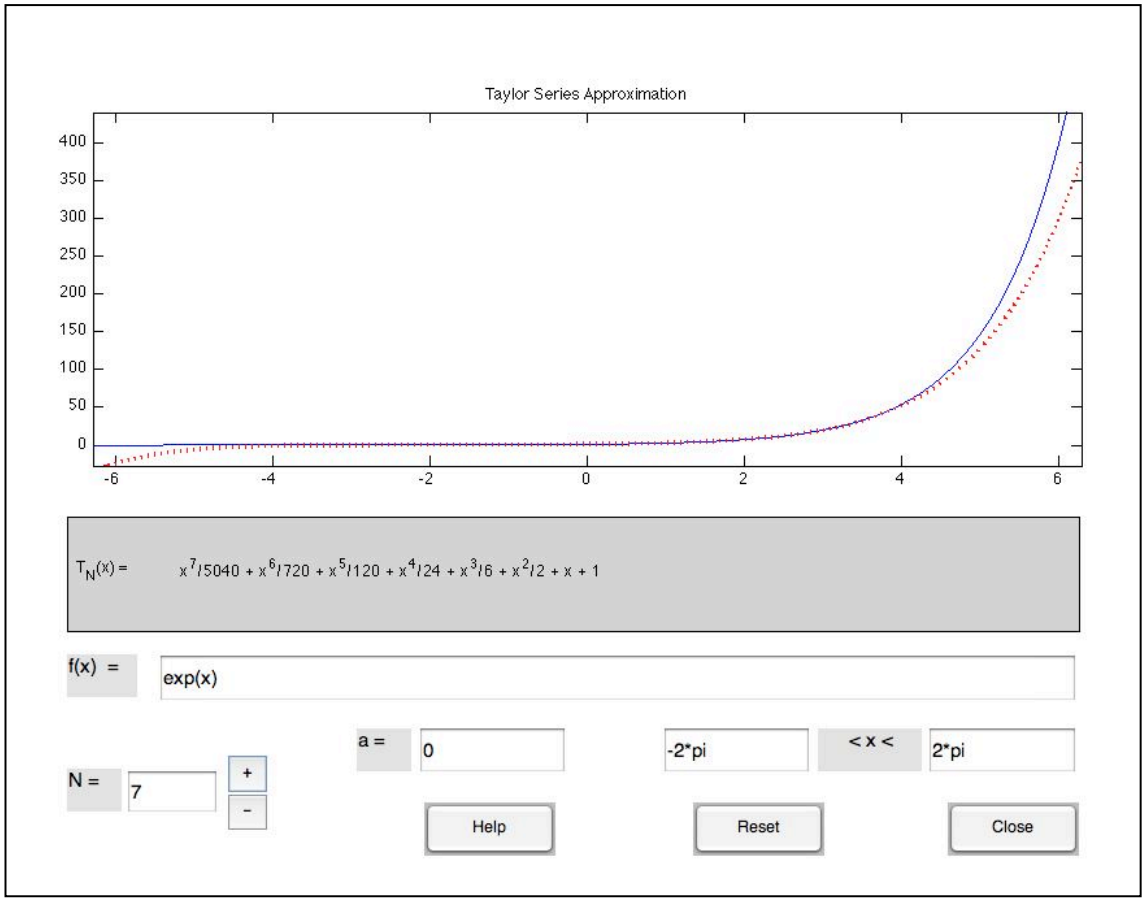

And this is good because we reduced the complexity of the function from exponential to polinomial!

The Taylor Series in Matlab

- Play with **`taylortool`** to get familiar Taylor Series

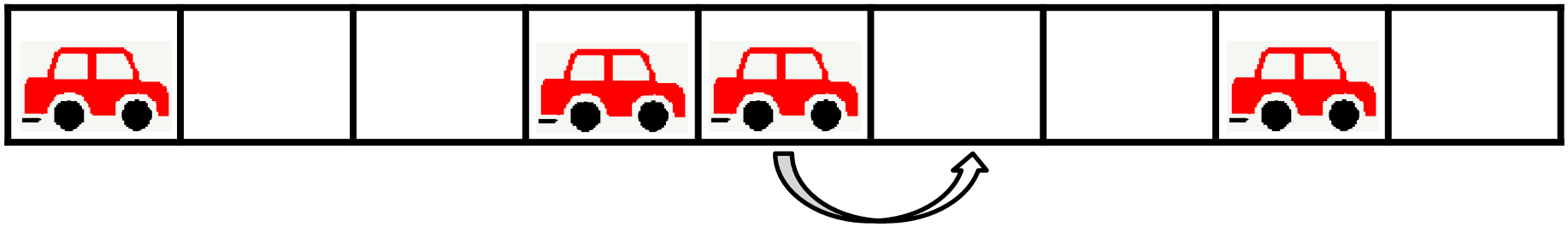


The Continuous Assumption

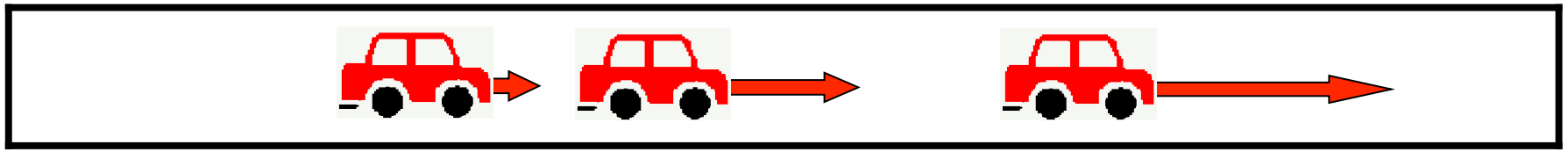
Resources: Google

From discrete to continuous space

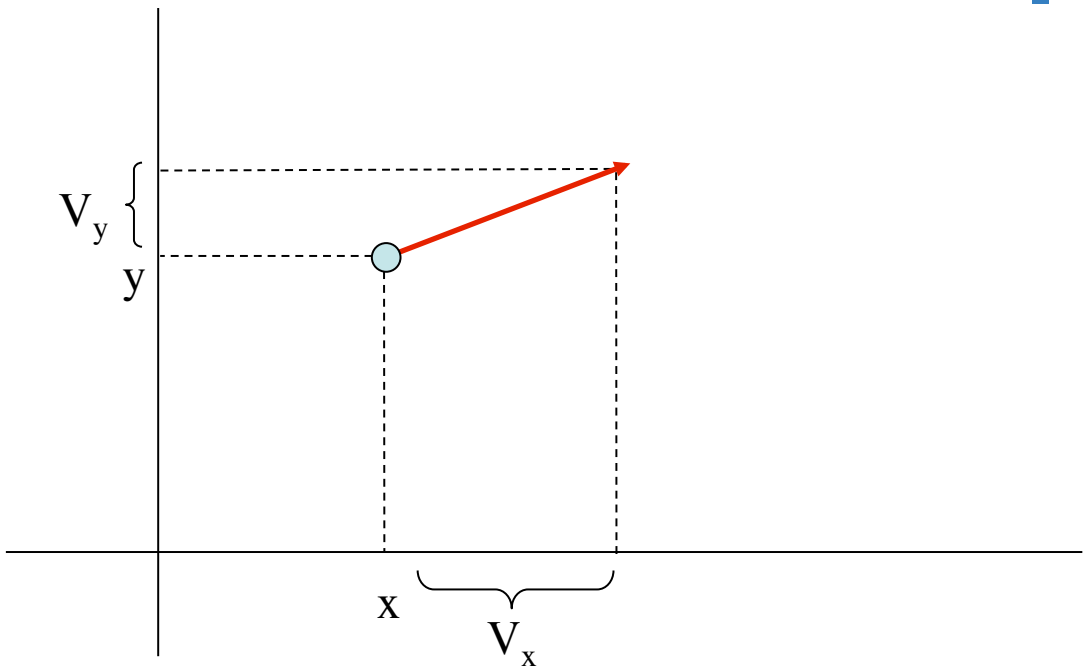
- Cellular automaton (CA)



- Continuous simulation



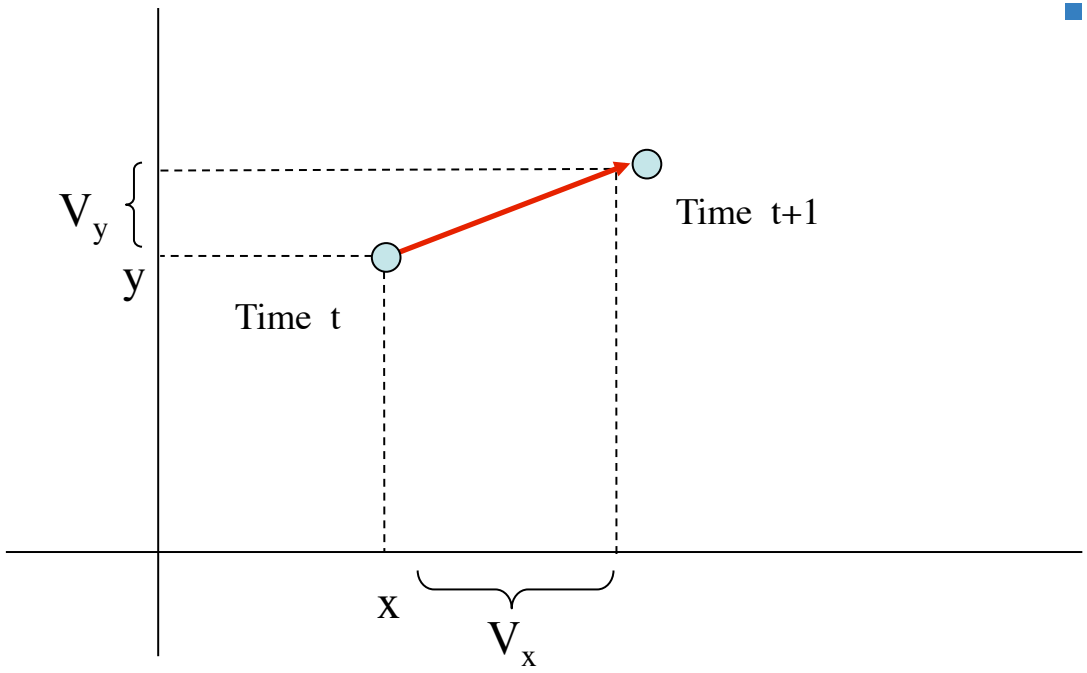
Modelling motion in continuous space



- Agents are characterized by:
  - position
    - X (m)
    - Y (m)
  - velocity
    - V_x (m/s)
    - V_y (m/s)

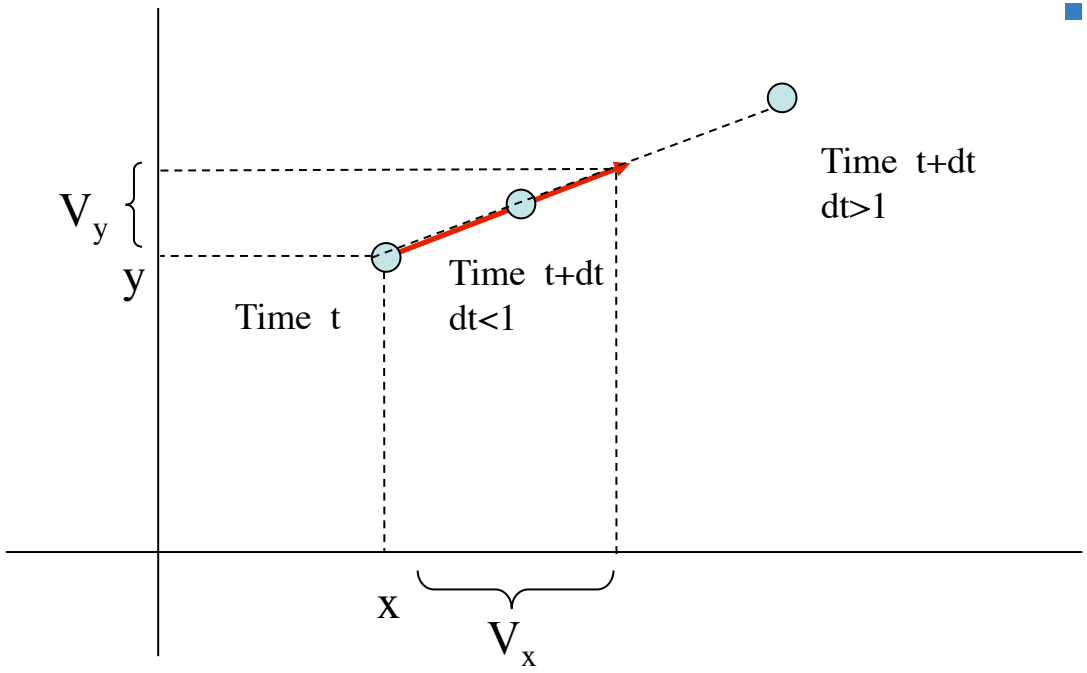
K. Donnay & S. Balietti / kdonnay@ethz.ch sbalietti@ethz.ch

Modelling motion in continuous space



- Next position after 1 second is given by:
  - $x(t+1) = x(t) + V_x(t)$
  - $y(t+1) = y(t) + V_y(t)$

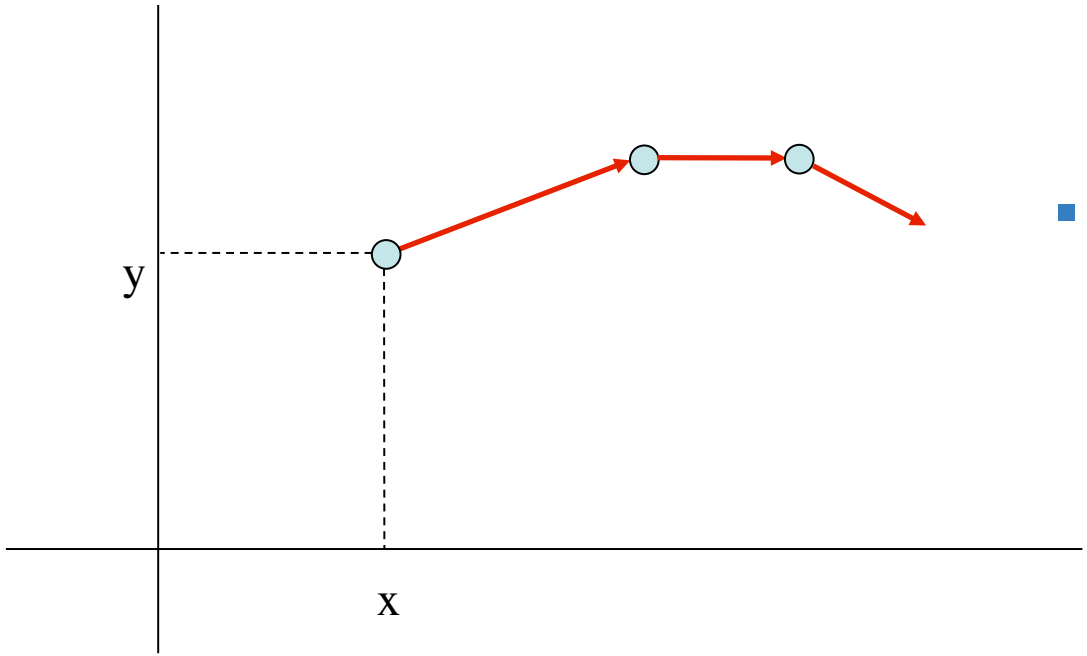
Modelling motion in continuous space



- Time step ***dt*** is often different from 1:
  - $x(t+dt) = x(t) + dt\ V_x(t)$
  - $y(t+dt) = y(t) + dt\ V_y(t)$

K. Donnay & S. Balietti / kdonnay@ethz.ch sbalietti@ethz.ch

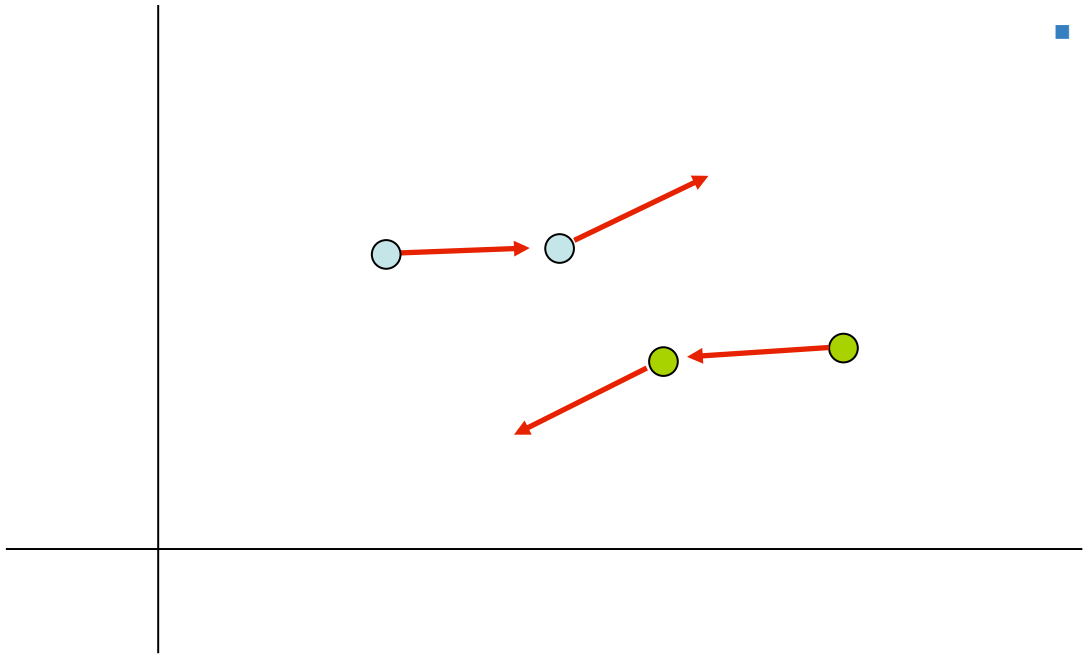
Modelling motion in continuous space



- The velocity is also updated in time.

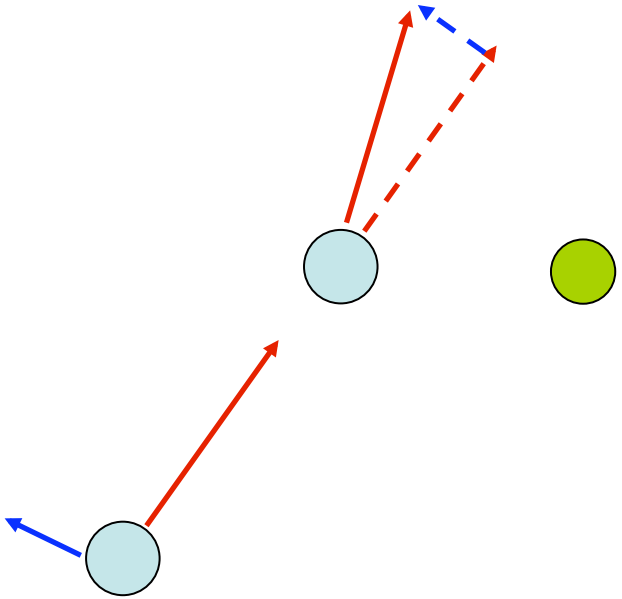
K. Donnay & S. Balietti / kdonnay@ethz.ch sbalietti@ethz.ch

Modelling motion in continuous space



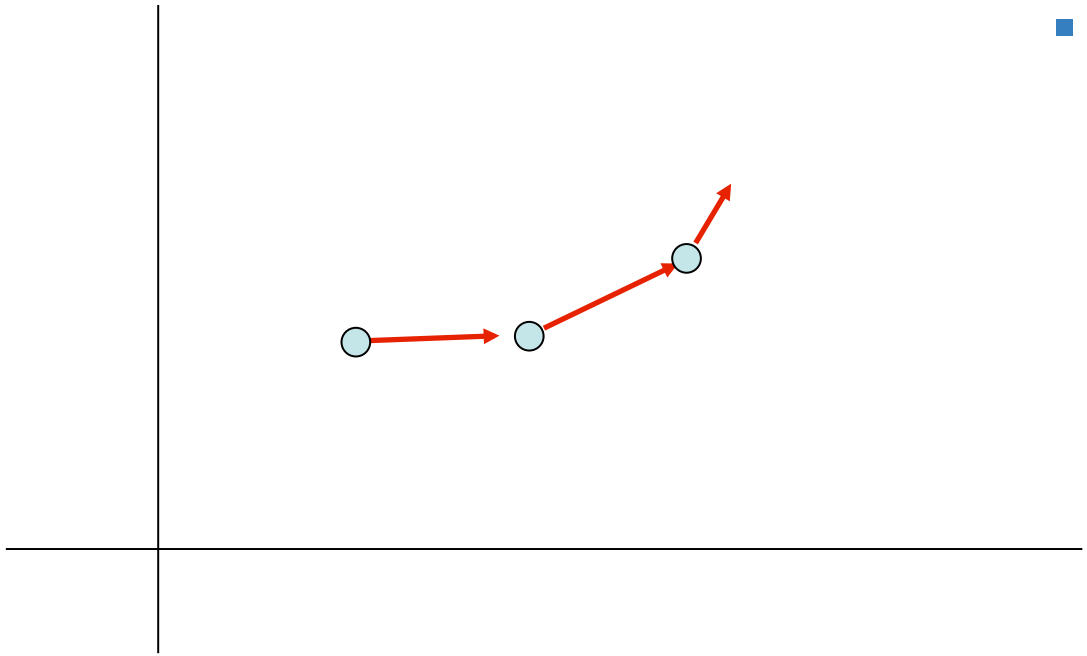
- The velocity changes as a result of inter-individual interactions
  - Repulsion
  - Attraction
  - Examples: atoms, planets, pedestrians, animals swarms…

Modelling motion in continuous space



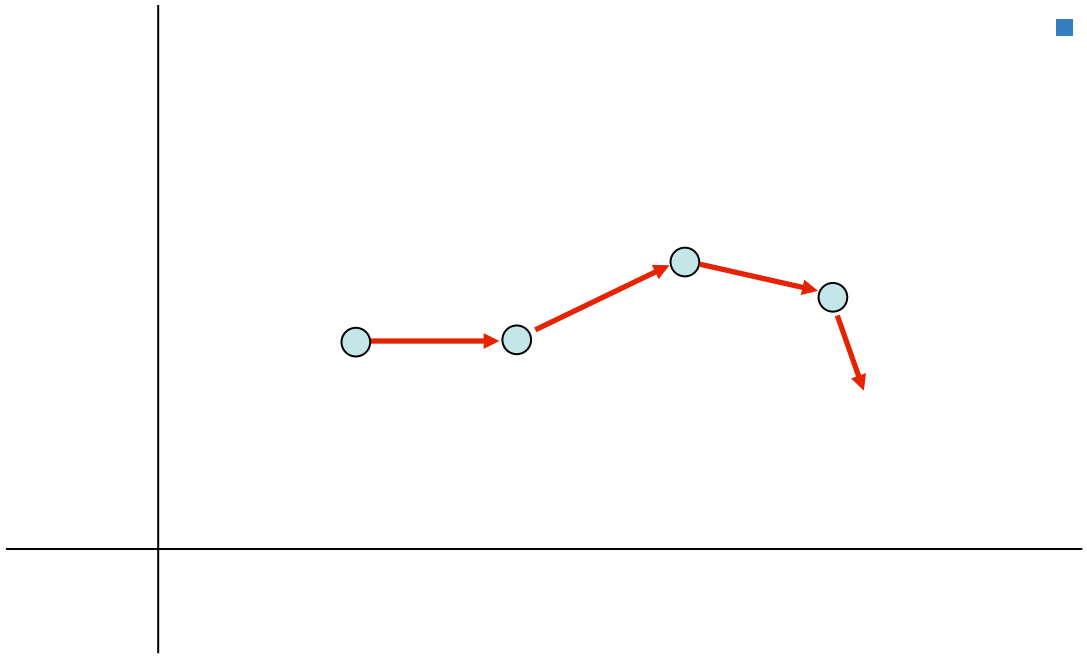
- The changes of velocity vector is defined by an acceleration.

Modelling motion in continuous space



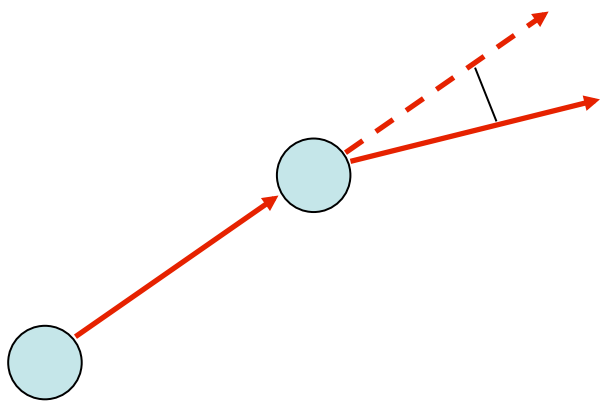
- The velocity changes as a result of interactions with the environment
  - Repulsion
  - Attraction
  - Examples: bacteria, ants, pedestrian trails …

Modelling motion in continuous space



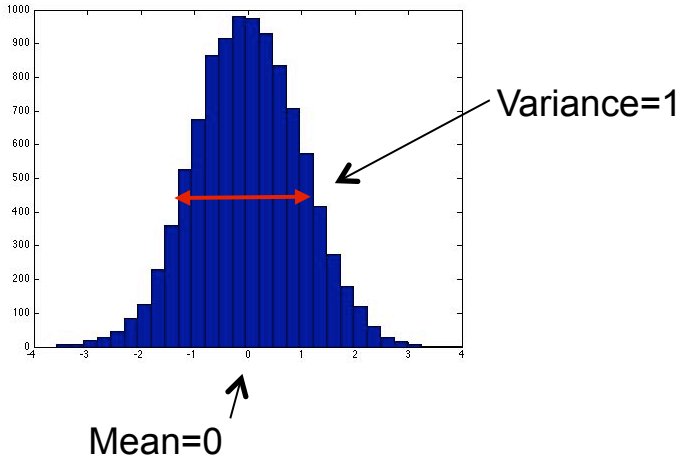
- The velocity changes as a result of a random process:
  - With or without bias
  - *Example*: exploration behaviour in many species of animals

Modelling motion in continuous space



- How to define a random move?
  - Choose a random angle in a normal distribution
  - Rotate the velocity by this angle

- In Matlab:
  - `newAngle = randn()`



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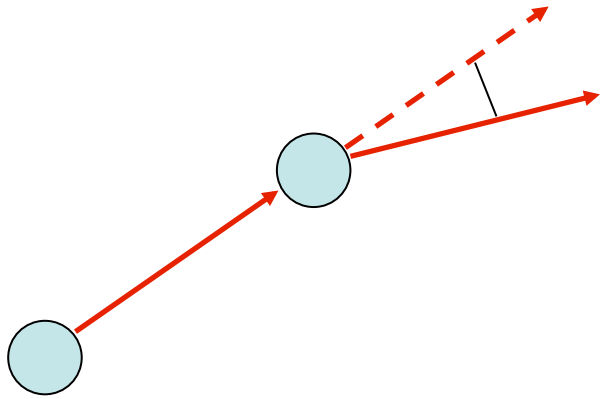
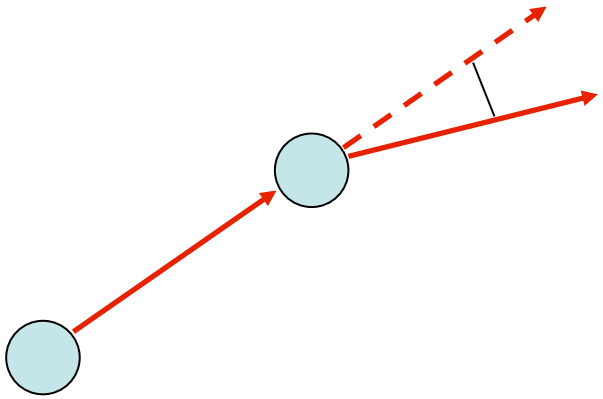
Modelling motion in continuous space



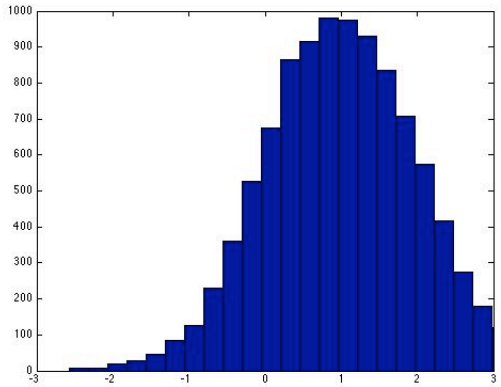
- How to define a random move?
  - Choose a random angle in a normal distribution
  - Rotate the velocity by this angle

- In Matlab:
  - `meanVal=2;`
  - `v=0.1;`
  - `newAngle = meanVal + v*randn()`

Modelling motion in continuous space



- How to define a BIASED random move?
  - Unbalance the distribution toward positive or negative values



- In Matlab:
  - `meanVal=2;`
  - `v=0.1;`
  - `newAngle = meanVal + v*r`

Programming a Continuous Simulator

- **Initialization**: Four elements are required to define the state of individuals

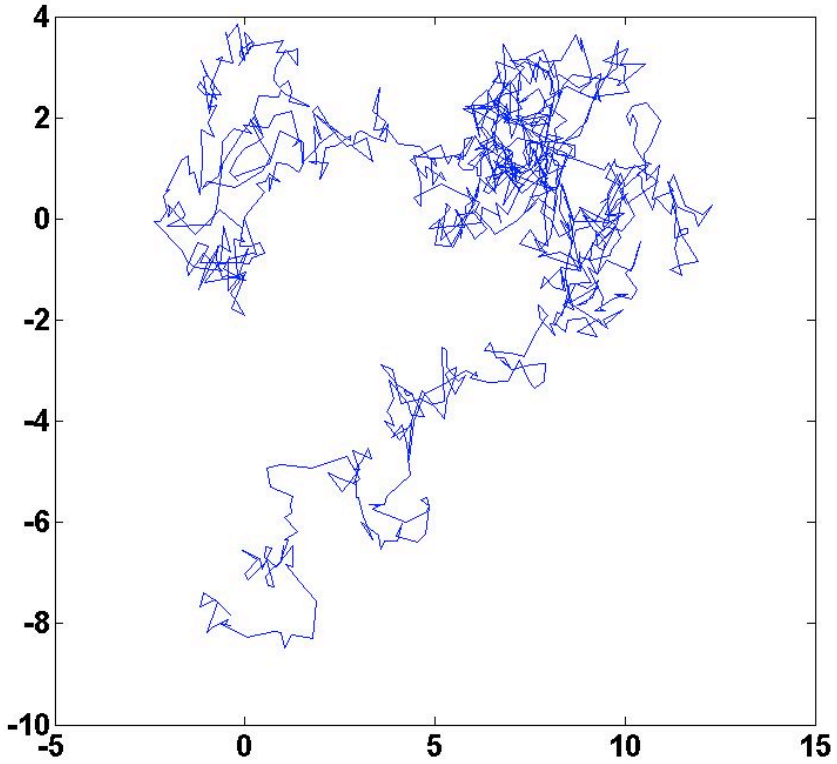
```
 X = [1 1 3 ];
 Y = [2 1 1];
Vx = [0 0 0.1];
Vy = [1 1 0.5];
```

- The **velocity** is updated first, and then the **position** as a function of the velocity and the **time step**

```
for t=1:dt:T
   for p=1:N
        [Vx(p) Vy(p)] = update( …) ;
        X(p) = X(p) + dt*Vx(p) ;
        Y(p) = Y(p) + dt*Vy(p) ;
   end
end
```

Brownian Motion

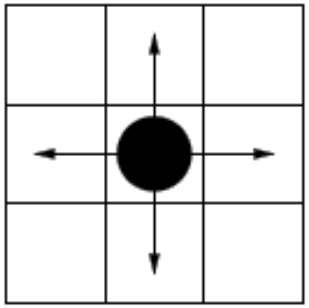
```
%Brownian Motion
t = 100;n = 1000;dt=t/n;
x = zeros(1,n);y = zeros(1,n);
dx = zeros(1,n);dy = zeros(1,n);
x(1) = 0;y(1) = 0;
hold on;
k = 1;
for j=2:n
    x(j) = x(j-1) + sqrt(dt)*randn;
    y(j) = y(j-1) + sqrt(dt)*randn;
    plot([x(j-1),x(j)],[y(j-1),y(j)]);
    M(k) = getframe;
    k = k + 1;
end
```



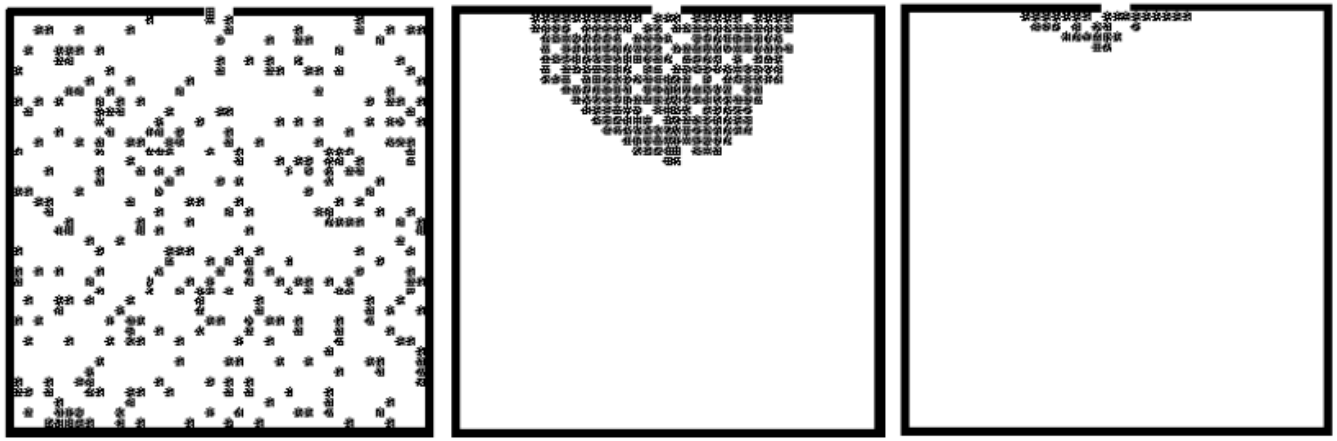
Pedestrian Simulation

- **CA model**
- **Many-particle model**

CA Model (A. Kirchner, A. Schadschneider, Phys. A, 2002)



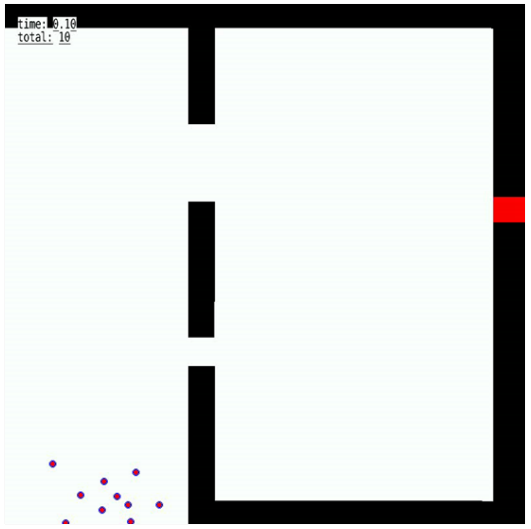
0	$p_{-1,0}$	0
$p_{0,-1}$	$p_{0,0}$	$p_{0,1}$
0	$p_{1,0}$	0

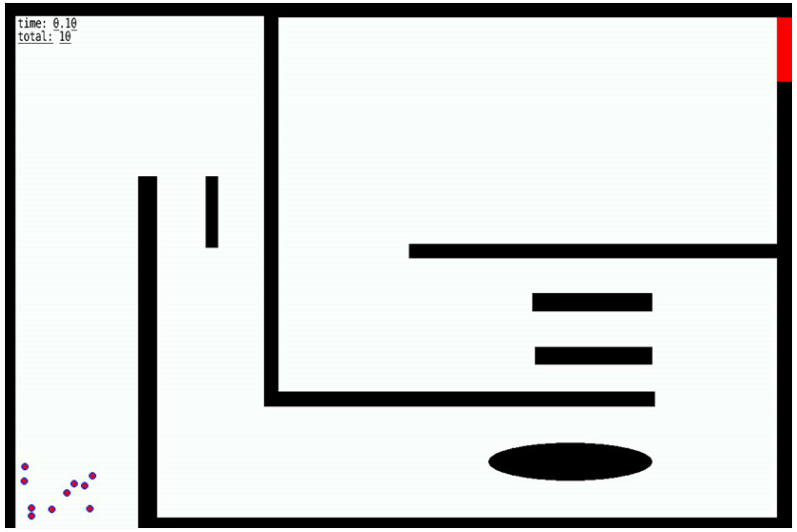


Many-particle Model (Helbing, Nature, 2000)

- Desired velocity
- Repulsive force

$$m_i \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = m_i \frac{v_i^0(t)\mathbf{e}_i^0(t) - \mathbf{v}_i(t)}{\tau_i} + \sum_{j(\neq i)} \mathbf{f}_{ij} + \sum_W \mathbf{f}_{iW}$$





Grids Vs Continuum Space

- Easier
- Implementation dependant
- More Limited
- One Element per Cell
- Computationally more intensive

- (more) Complex
- Elegant
- Extremely Powerful
- Agents Floats in Space
- Computationally faster (with analytical derivates)

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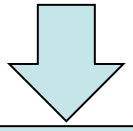

You can write a good paper with both approaches if you have a good idea

Projects

- Today, there are no exercises.
- Please check out the additional material on the website regarding Taylor Series, and stochastic processes.

- You can work on your projects and we will supervise you.

References

- http://en.wikipedia.org/wiki/Taylor_expansion
- http://en.wikipedia.org/wiki/Random_walk
- http://www.mathworks.com/matlabcentral/fileexchange/22003
- *Nature* **407**, 487-490 (2000) **Simulating dynamical features of escape panic.** Dirk Helbing, Illés Farkas and Tamás Vicsek